Energy dissipation caused by boundary layer instability at vanishing viscosity

Natacha Nguyen van yen, Marie Farge, *ENS Paris,*
Kai Schneider, *Aix-Marseille Université,*
Matthias Waidmann and Rupert Klein,
Freie Universität, Berlin,

Workshop on 'Mathematical and Computational Problems of Incompressible Fluid Dynamics'
IMPA, Rio de Janeiro, August 10th 2018

What is the inviscid limit of Navier-Stokes?

Navier-Stokes equations with
no-slip boundary conditions:

$$\begin{cases} \partial_t \mathbf{u} + (\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u} = -\boldsymbol{\nabla} p + \frac{1}{\mathrm{Re}}\boldsymbol{\nabla}^2\mathbf{u} \\ \boldsymbol{\nabla}\cdot\mathbf{u} = 0 \\ \mathbf{u}_{|\partial\Omega} = \mathbf{0}, \qquad \mathbf{u}(0,\cdot) = \mathbf{v} \end{cases} \longrightarrow \mathbf{u}_{\mathrm{Re}}(t,\mathbf{x}) \quad \text{for } \nu \to 0,\ \mathrm{Re} \to +\infty$$

Re = $VL\nu^{-1}$ the Reynolds number

Same initial conditions — ?

Euler equations with
slip boundary conditions:

$$\begin{cases} \partial_t \mathbf{u} + (\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u} = -\boldsymbol{\nabla} p \\ \boldsymbol{\nabla}\cdot\mathbf{u} = 0 \\ \mathbf{u}_{|\partial\Omega}\cdot\mathbf{n} = \mathbf{0}, \qquad \mathbf{u}(0,\cdot) = \mathbf{v} \end{cases} \longrightarrow \mathbf{u}(t,\mathbf{x}) \quad \text{for } \nu = 0,\ \mathrm{Re} = +\infty$$

Leonhard Euler
(1707-1783)

Jean Le Rond d'Alembert
(1717-1783)

1750: Euler's problem

On 16 May 1748 Euler, president of the Prussian Academy of Sciences, read the problem he proposed for the Prize of Mathematics to be given in 1750 :

> *'Deduce from new principles, as simple as possible,*
> *a theory to explain the resistance*
> *exerted on a body moving in a fluid,*
> *as a function of the body's velocity, shape and mass,*
> *and of the fluid's density and compressibility'.*

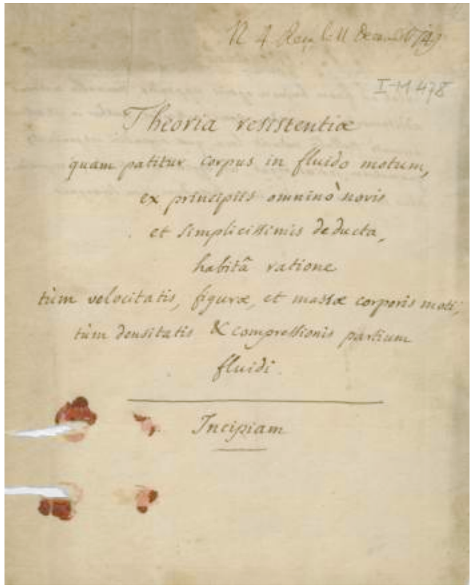
Six mathematicians, including d'Alembert, sent a manuscript, but Euler was not satisfied with them and decided to postpone the prize to 1752.

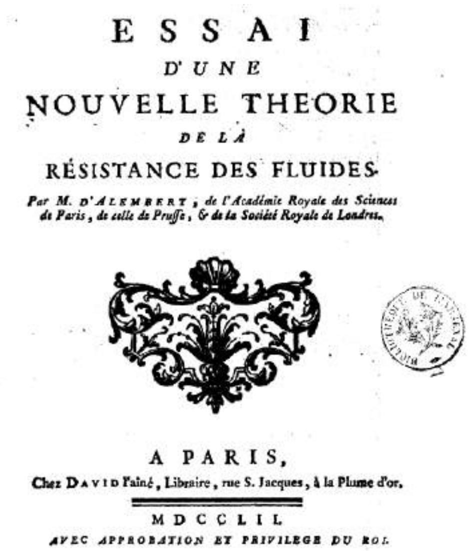
Grimberg, *D'Alembert et les équations aux dérivées partielles en hydrodynamique, Thèse de Doctorat, Université de Paris VII, 1998*

1752: d'Alembert's paradox

D'Alembert was upset and took back his manuscript of 1749, translated it into French and published it in 1752.

1749



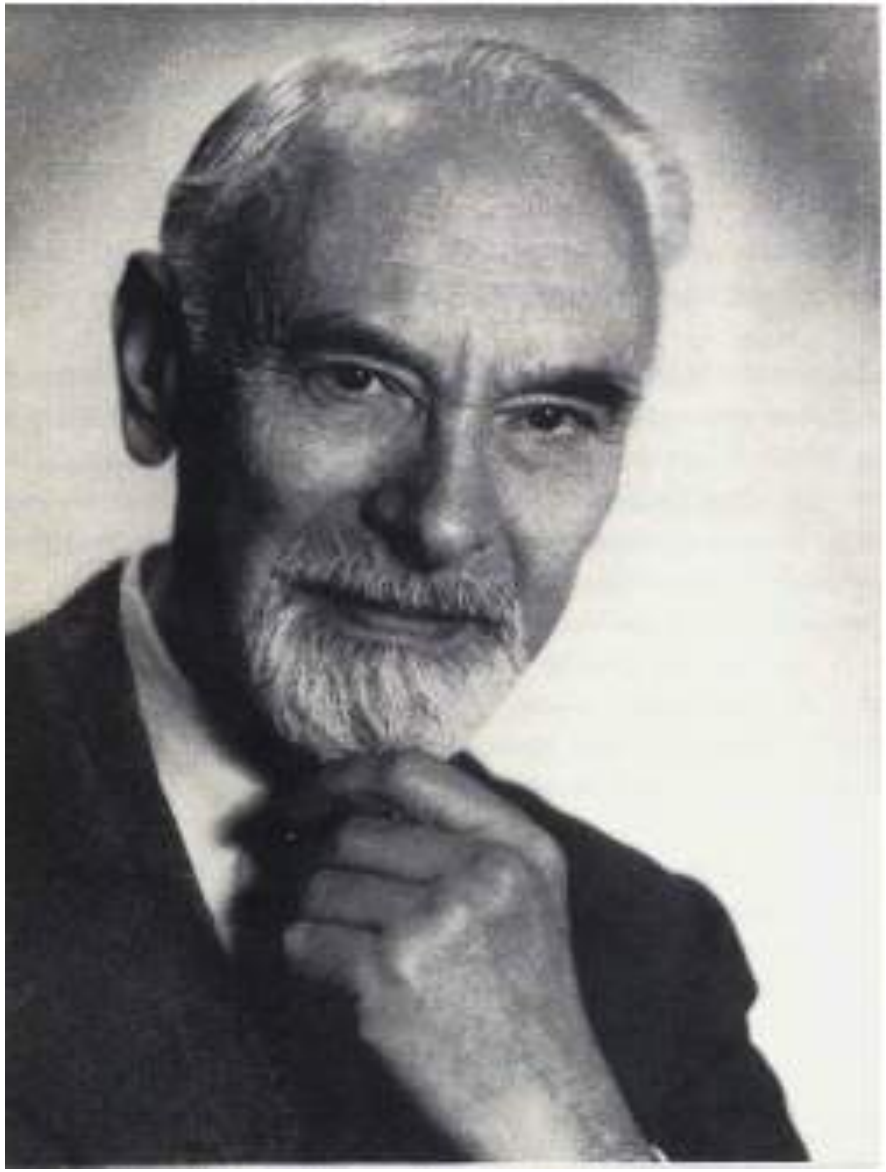


1752

'It seems to me that the theory, developed in all possible rigor, gives, at least in several cases, a strictly vanishing resistance, a singular paradox which I leave to future geometers to elucidate.'

https://gallica.bnf.fr/ark:/12148/bpt6k206036b

Ludwig Prandtl
(1875-1953)

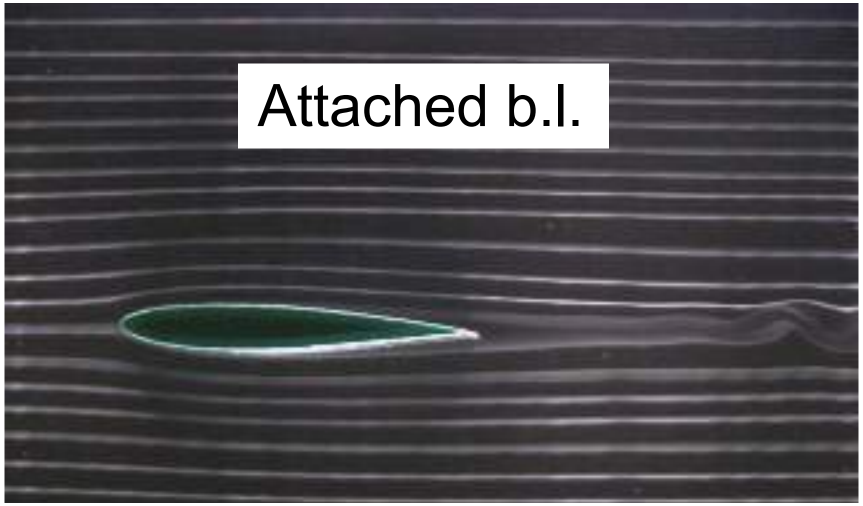


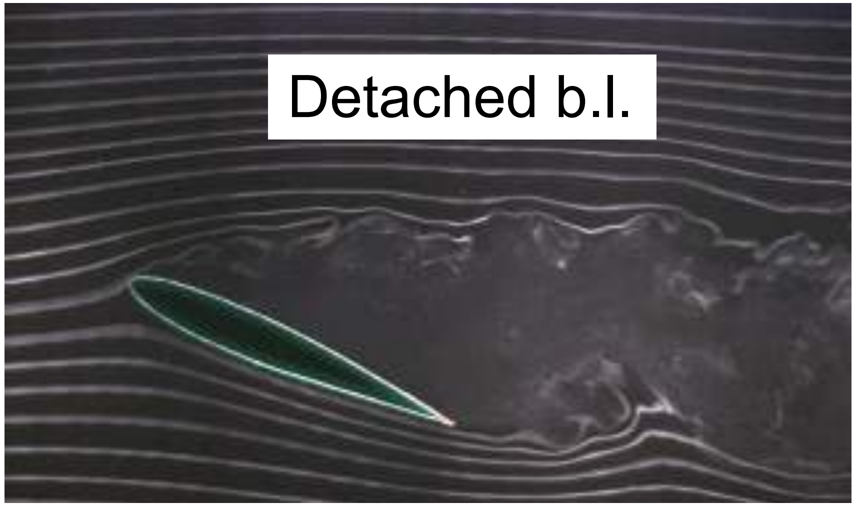
Toshio Kato
(1917-1999)

1904: Prandtl's boundary layer theory

- Prandtl (1904) predicted that the thickness of the boundary layer in contact with a solid body *(left)* scales as $Re^{-1/2}$,

  the inverse square root of the Reynolds number *Re*,
- But Prandtl's theory does not apply for separated flow regions where the boundary layer detaches from the solid body *(right)*.





Prandtl, Über Flüssigkeitsbewegung bei sehr kleiner Reibung, Proceedings of ICM in Heidelberg, 484-491, 1904

1984: Kato's theorem

Navier-Stokes solution converges towards the Euler solution, if and only if, energy dissipation vanishes

$$\Delta E_{\mathrm{Re}}(0,T) = \mathrm{Re}^{-1} \int_0^T \mathrm{d}t \int_\Omega \mathrm{d}\mathbf{x} \left|\nabla \mathbf{u}(t,\mathbf{x})\right|^2 \underset{\mathrm{Re}\to\infty}{\longrightarrow} 0,$$

$$\nu \to 0$$

and, if and only if, this happens in a boundary layer of thickness inversely proportional to the Reynolds number *Re*

Kato, 1984, Remarks on zero viscosity limit for non stationary Navier-Stokes flows with boundary, MSRI Berkeley

~~$\delta x \propto \mathrm{Re}^{-\frac{1}{2}}$~~

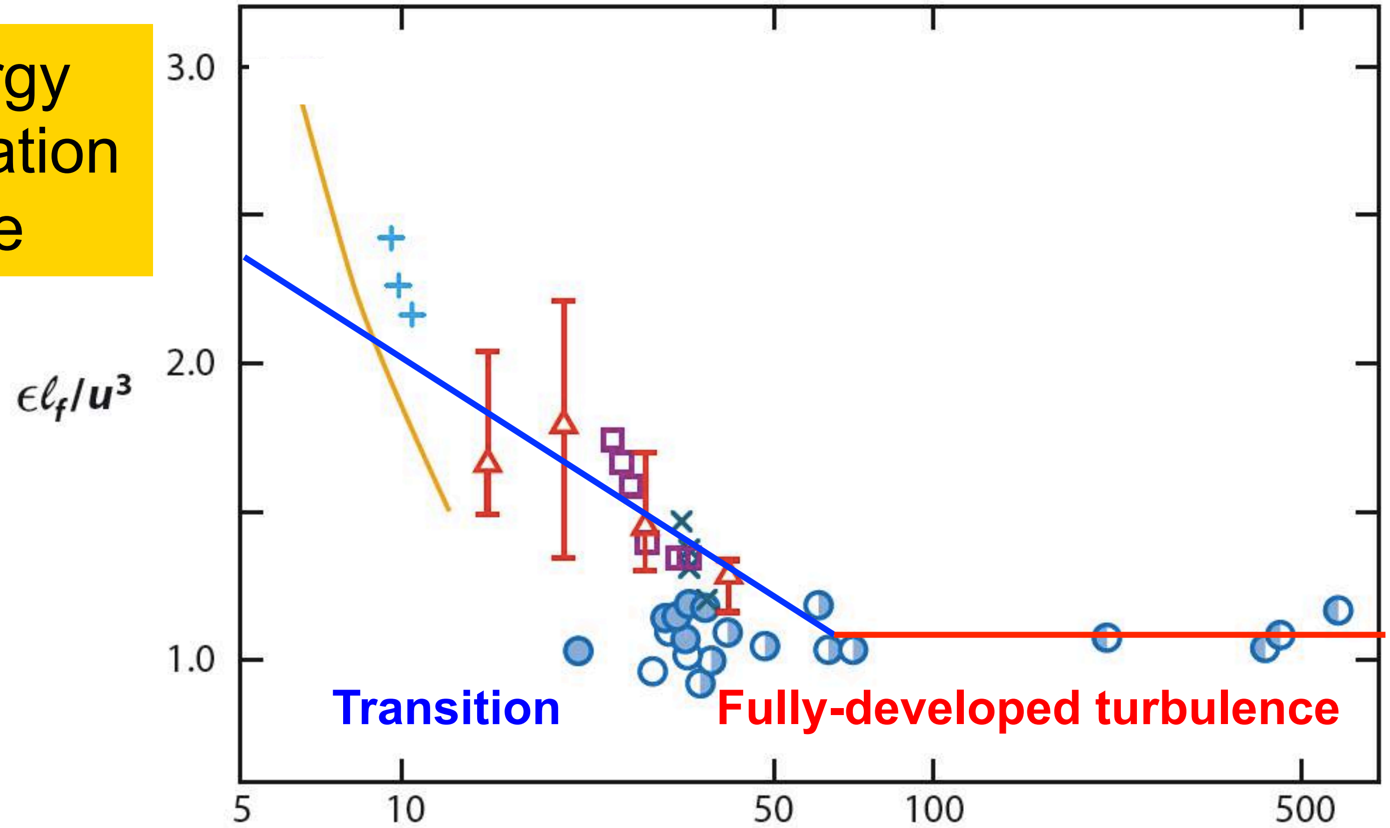
$$\delta x \propto \mathrm{Re}^{-1}$$

This requires using smaller resolution to compute high Reynolds flows than predicted by Prandtl's theory

Laboratory experiments

*Vassilicos, Ann. Rev. Fluid Mech., **47**, 2015*

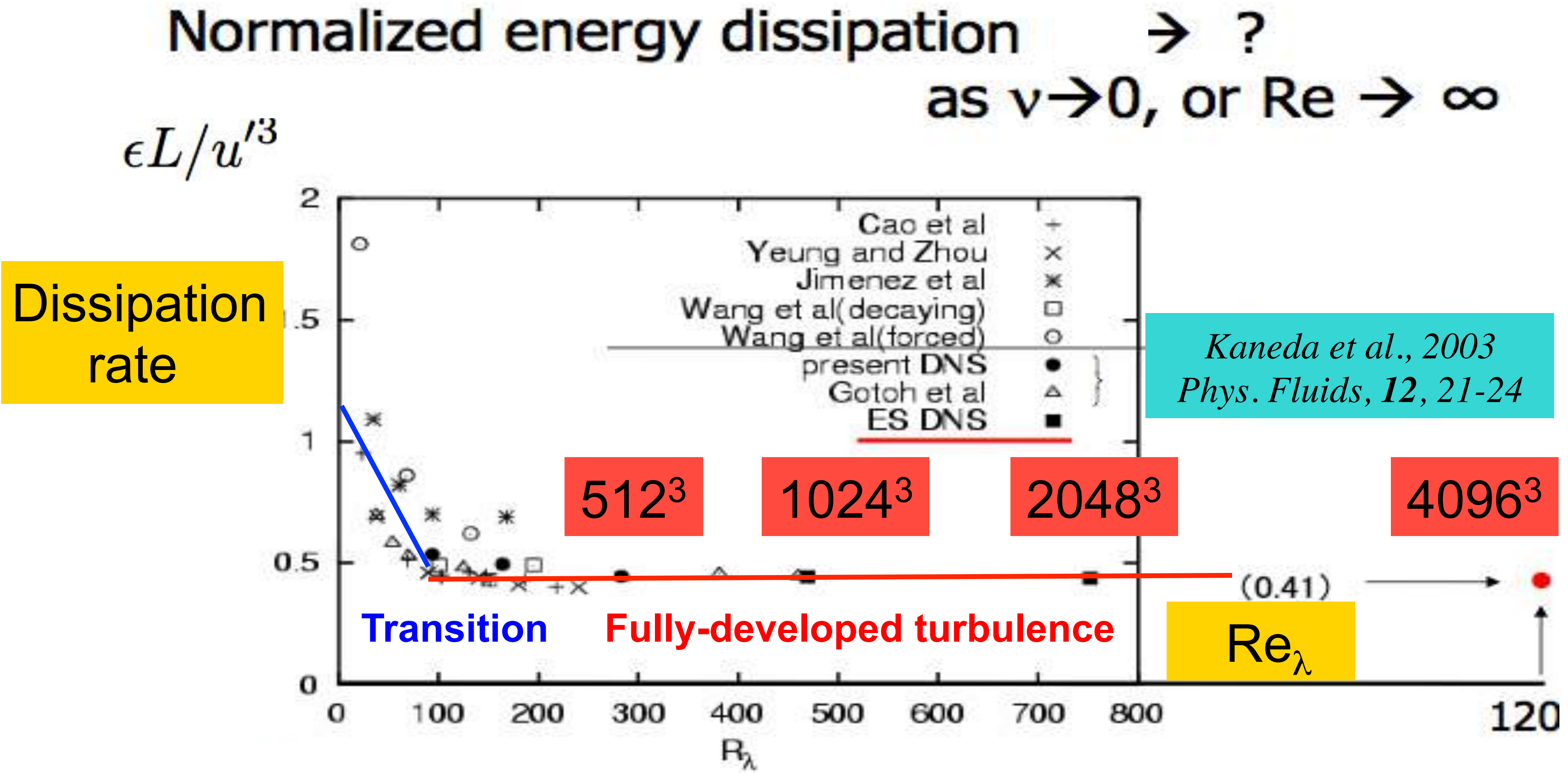
Energy dissipation rate



$Re_\lambda = Re^{1/2}$

For $\nu \rightarrow 0$ or $\mathrm{Re} \rightarrow +\infty$
energy dissipation does not vanish
but becomes constant

Numerical experiments



Both laboratory and numerical experiments
show that the dissipation rate of turbulent flows
becomes independent of the fluid viscosity for large *Re*

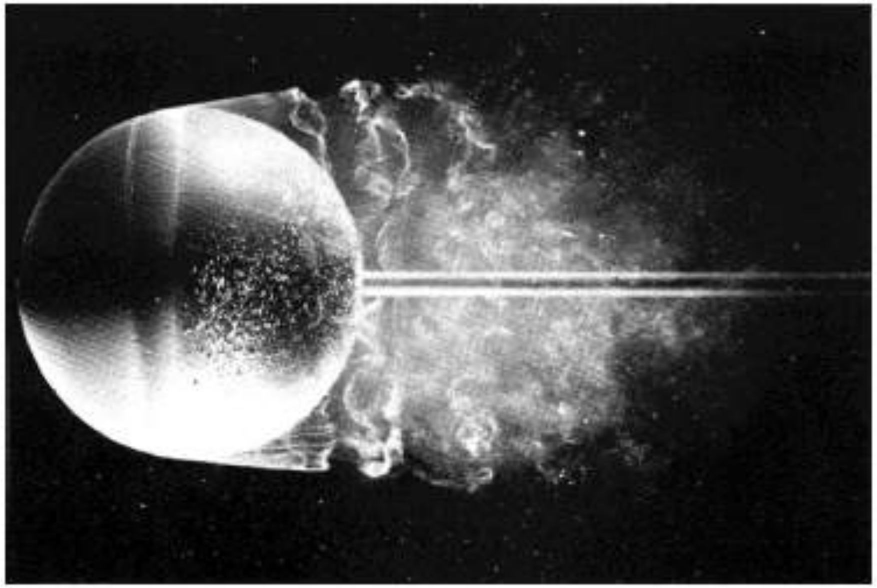
Dissipation of energy in the inviscid limit

- In an incompressible flow ($\rho = 1$)

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\int \frac{\mathbf{u}^2}{2} = -\nu \int \omega^2 = -2\nu Z$$

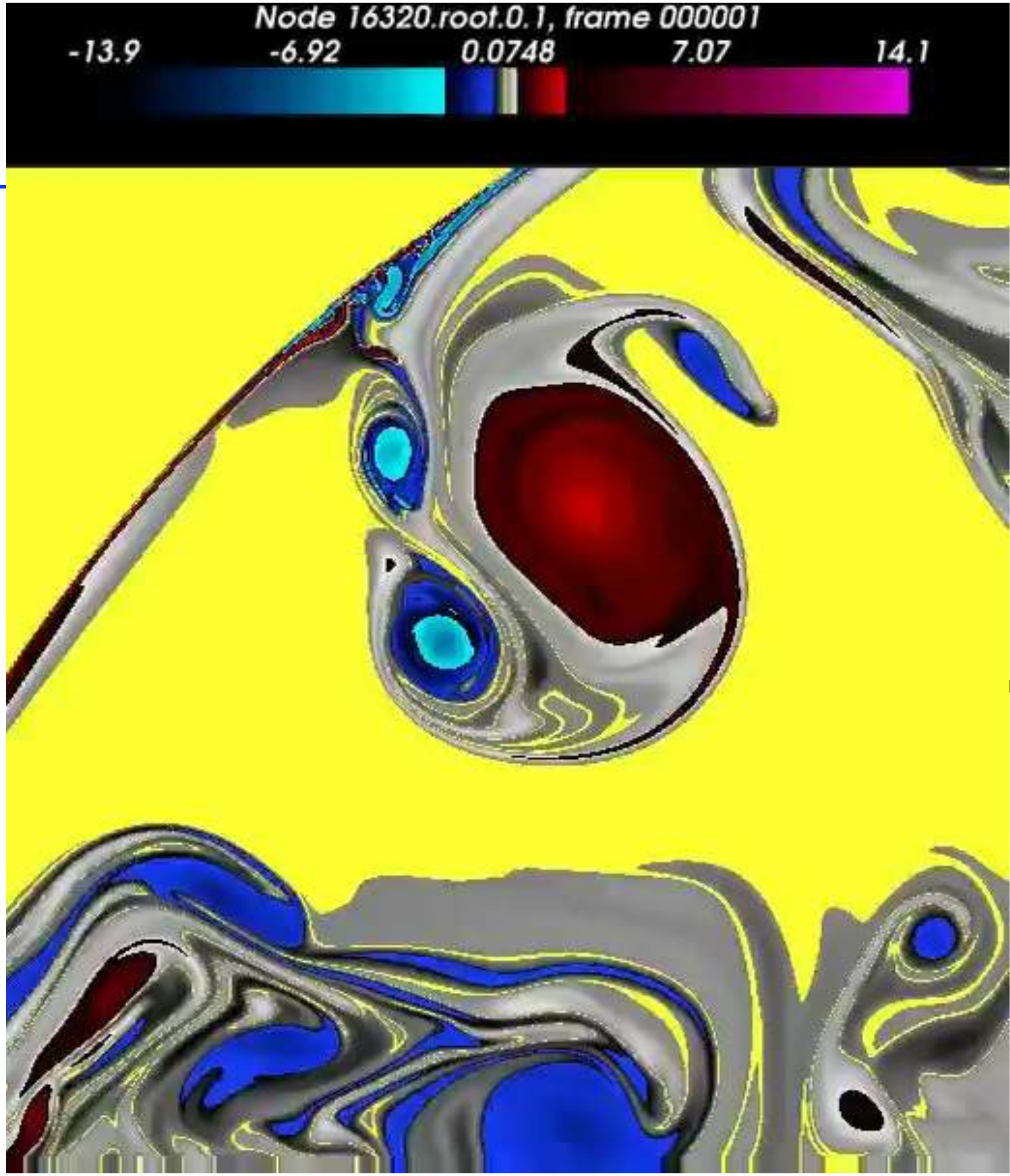
- To dissipate energy, vorticity needs to be created and/or amplified, in such a way that $Z \sim \nu^{-1}$.

Possible vorticity distributions:
$\omega \sim \nu^{-1/2}$ over $O(1)$ area,
$\omega \sim \nu^{-1}$ over $O(\nu)$ area.



with E energy, Z enstrophy,
ν fluid kinematic viscosity,
ω flow vorticity.

Resolution
$N=8192^2$

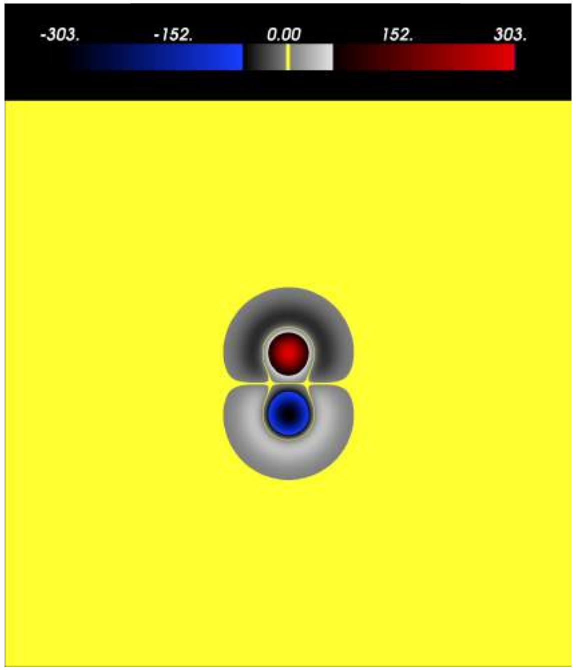


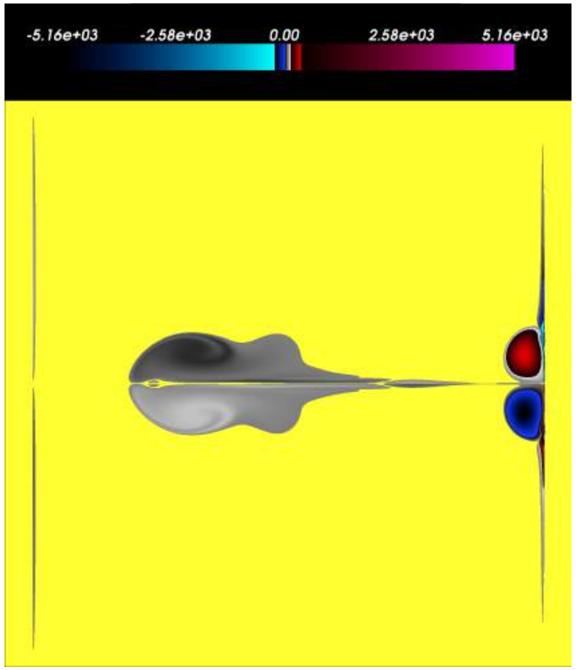
Time evolution of vorticity at the wall
computed on IBM Blue-Gene, IDRIS, 2010 (100 Tflops)

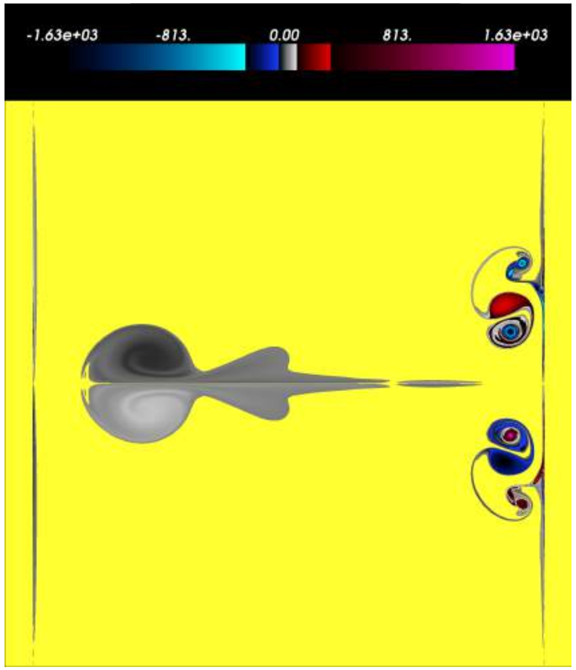
Nguyen van yen, M. F. and Schneider, 2010

Dipole crashing onto a plane wall

DNS
Resolution
N=8192^2





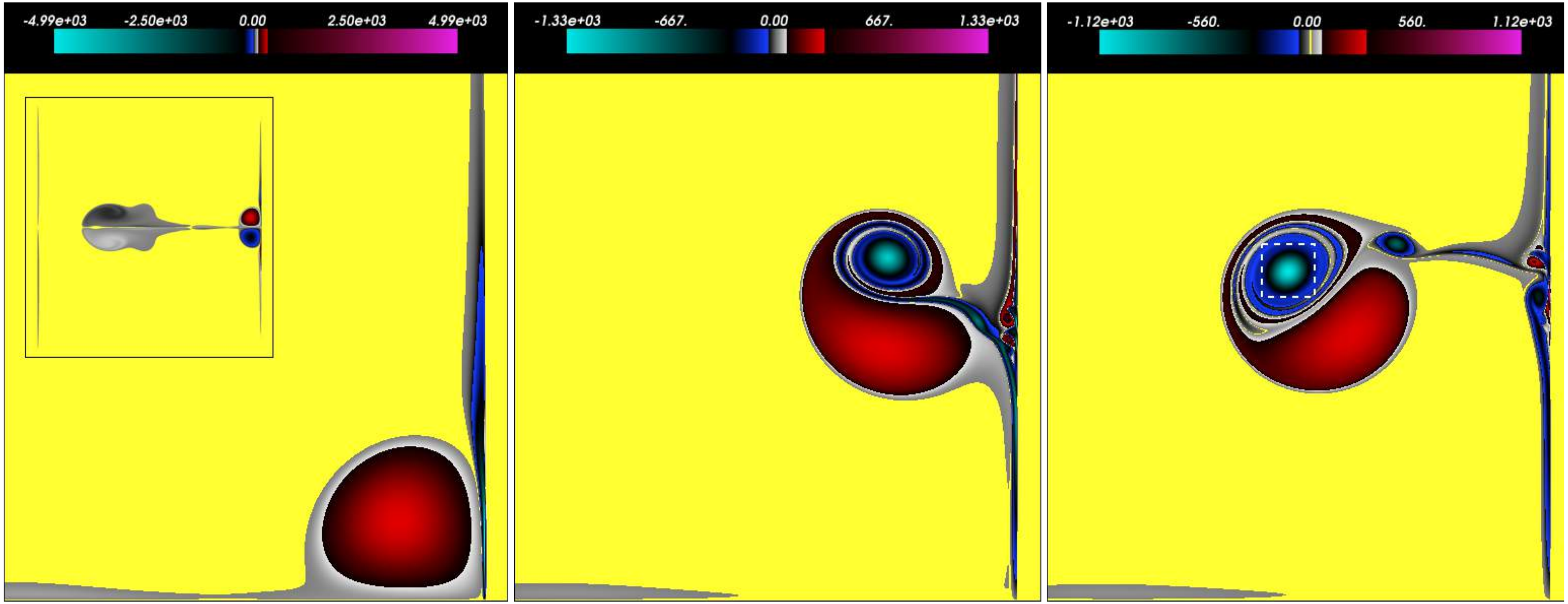


Dipole crashing onto a wall in 2D

Resolution
N=16384^2

Navier-Stokes equations with volume penalization integrated using Fourier

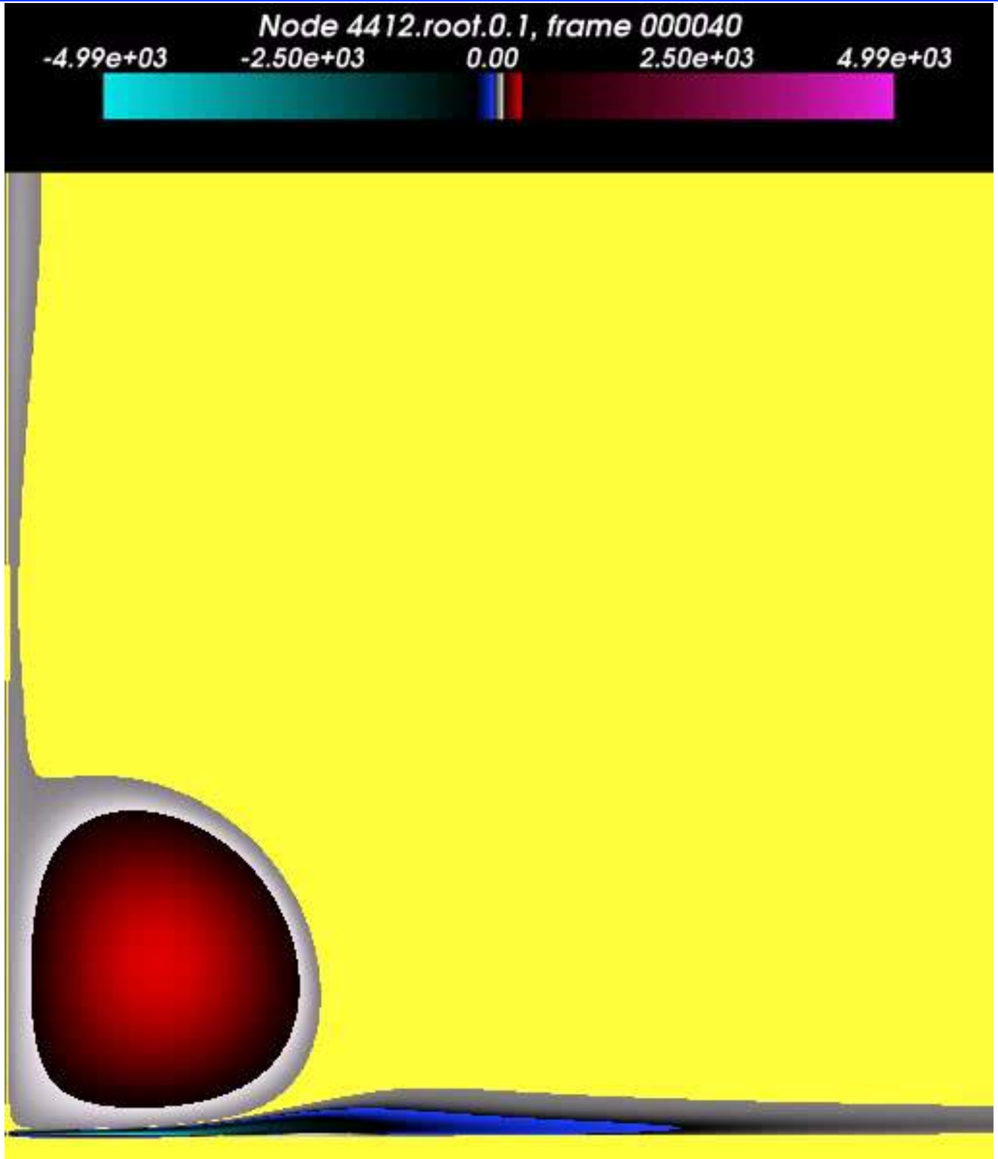
*Nguyen van yen, M. F. and Schneider, PRL, **106**(18), 2011*



t=0.3 *t=0.4* *t=0.5*

Dipole crashing onto a wall

DNS
Resolution
N=8192²



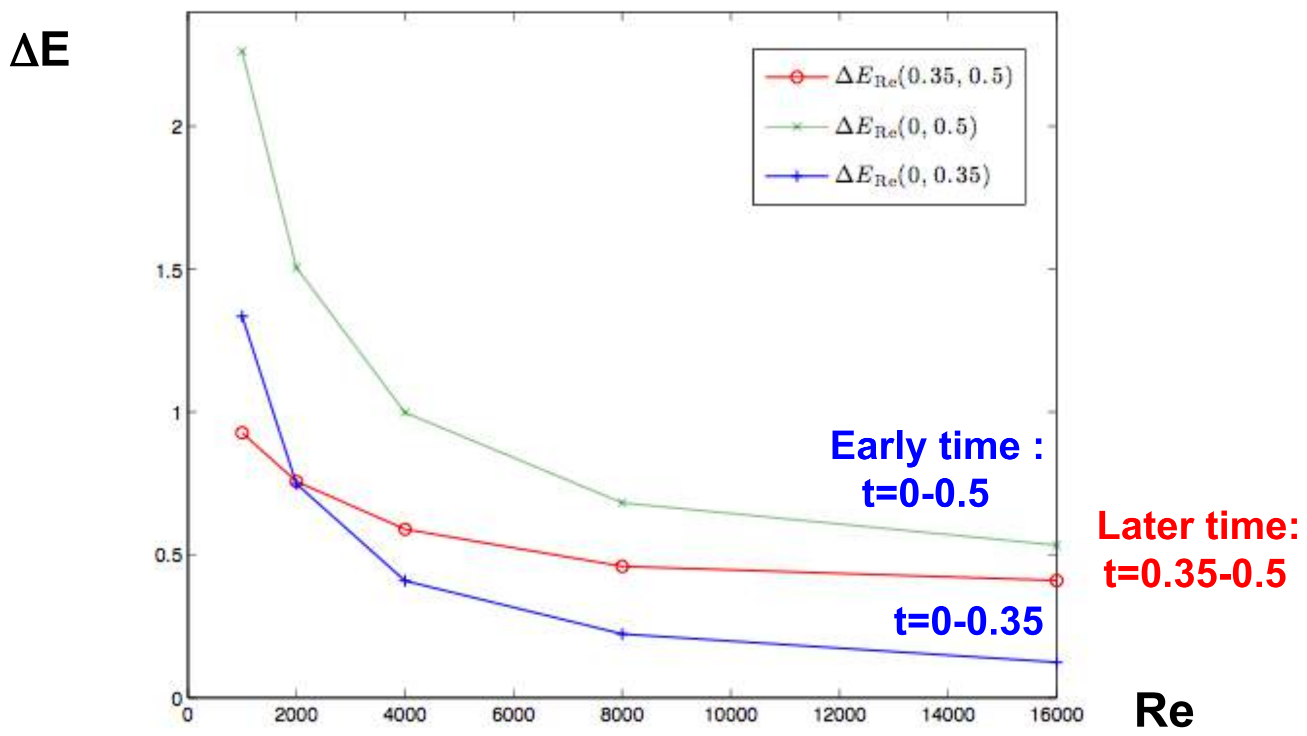
Re=8000

Production of vortices where boundary layer detaches

*Nguyen van yen, M. F. and Schneider, PRL, **106**(18), 2011*

Energy dissipation

Energy dissipated
when the dipole crashes onto the wall
at increasing Reynolds numbers



32. Dissipative structures

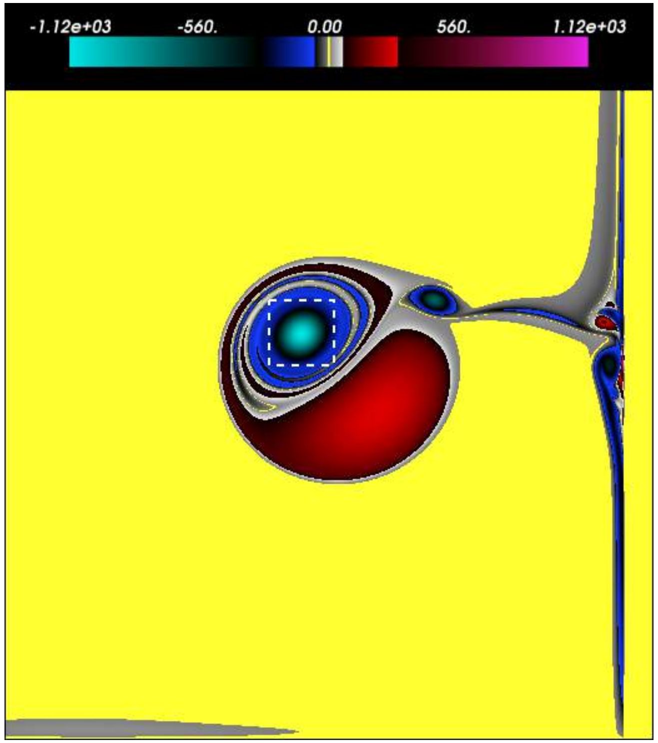
- Our experiments suggest that the flow remains dissipative in the inviscid limit,
- it is tempting to relate the observed structures to energy dissipation,
- the kinetic energy density $e = \frac{|\mathbf{u}|^2}{2}$ obeys:

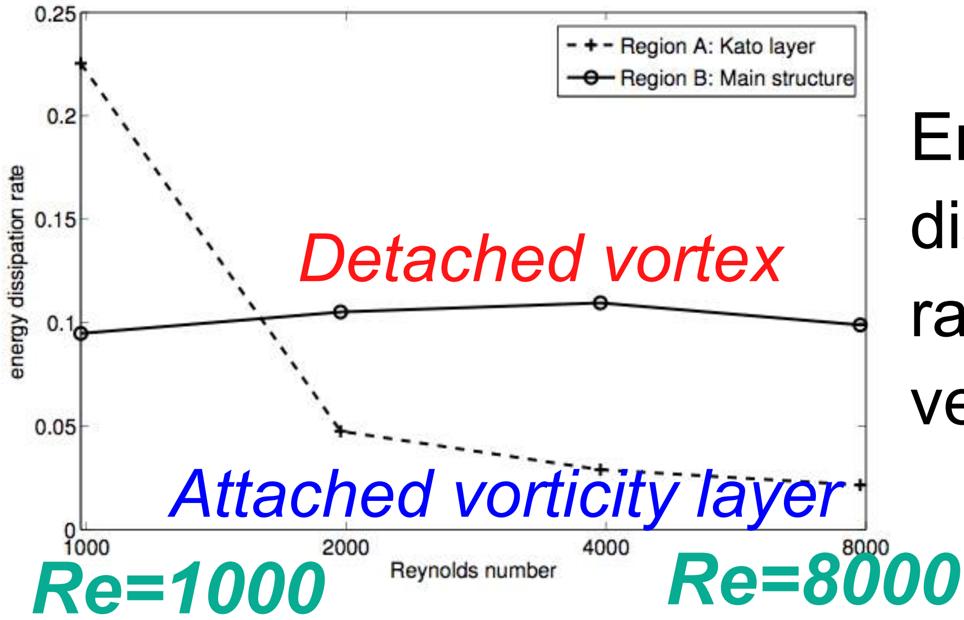
$$\partial_t e + \mathbf{u} \cdot \nabla(e + p) = \nu \Delta e - \nu |\nabla \mathbf{u}|^2$$

Local dissipation rate

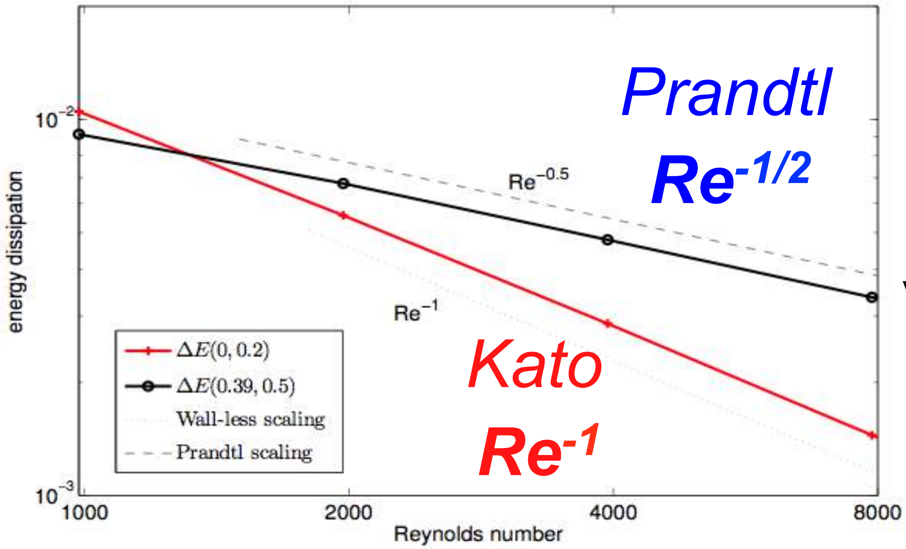
Production of dissipative structures

*Nguyen van yen, M. F. and Schneider, PRL, **106**(18), 2011*



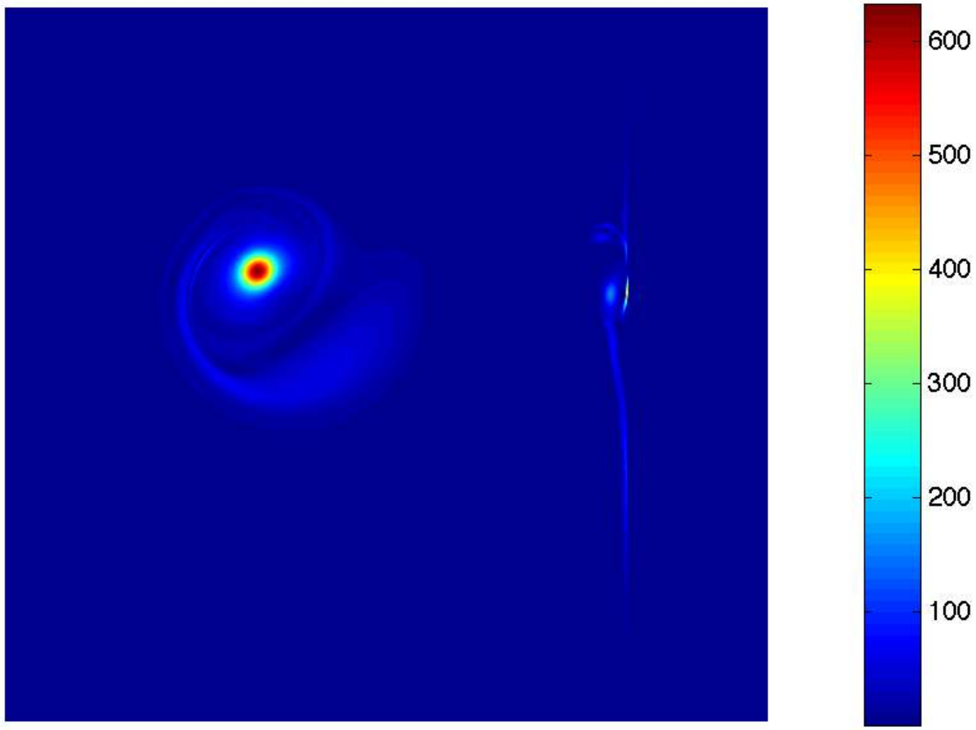


Energy dissipation rate $(-2\nu Z)$ versus *Re*



Energy Dissipation versus *Re*

Local dissipation rate

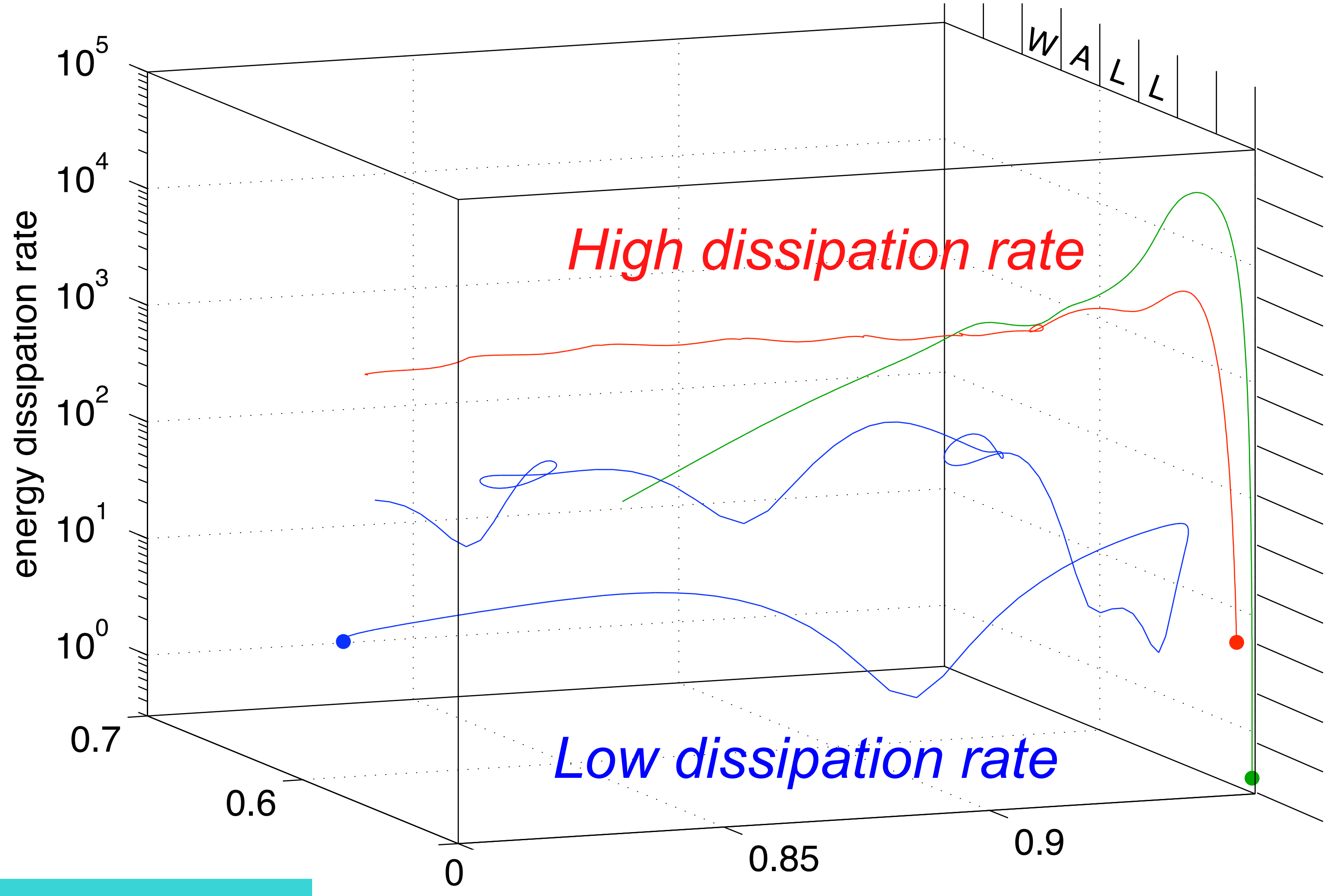


Local dissipation rate
for the dipole-wall collision
at t= 0.5

The strongest values of the energy dissipation rate is observed inside the main vortex that detached from the boundary layer,

rather than inside the boundary layer itself.

Production of dissipative structures



*R. Nguyen van yen, M. F. and K. Schneider, PRL, **106**(18), 2011*

2011

PHYSICAL REVIEW LETTERS

Energy Dissipating Structures Produced by Walls in Two-Dimensional Flows at Vanishing Viscosity

Romain Nguyen van yen and Marie Farge
LMD-CNRS-IPSL, École Normale Supérieure, Paris, France

Kai Schneider
M2P2-CNRS and CMI, Université d'Aix-Marseille, Marseille, France
(Received 13 October 2010; published 3 May 2011)

2013

PHYSICS OF FLUIDS **25**, 093104 (2013)

The effect of slip length on vortex rebound from a rigid boundary

D. Sutherland,[1,a)] C. Macaskill,[1] and D. G. Dritschel[2]
[1]*School of Mathematics and Statistics, University of Sydney, Sydney 2006, Australia*
[2]*School of Mathematics and Statistics, University of St. Andrews, St. Andrews KY16 9SS, United Kingdom*

(Received 22 May 2013; accepted 16 August 2013; published online 23 September 2013)

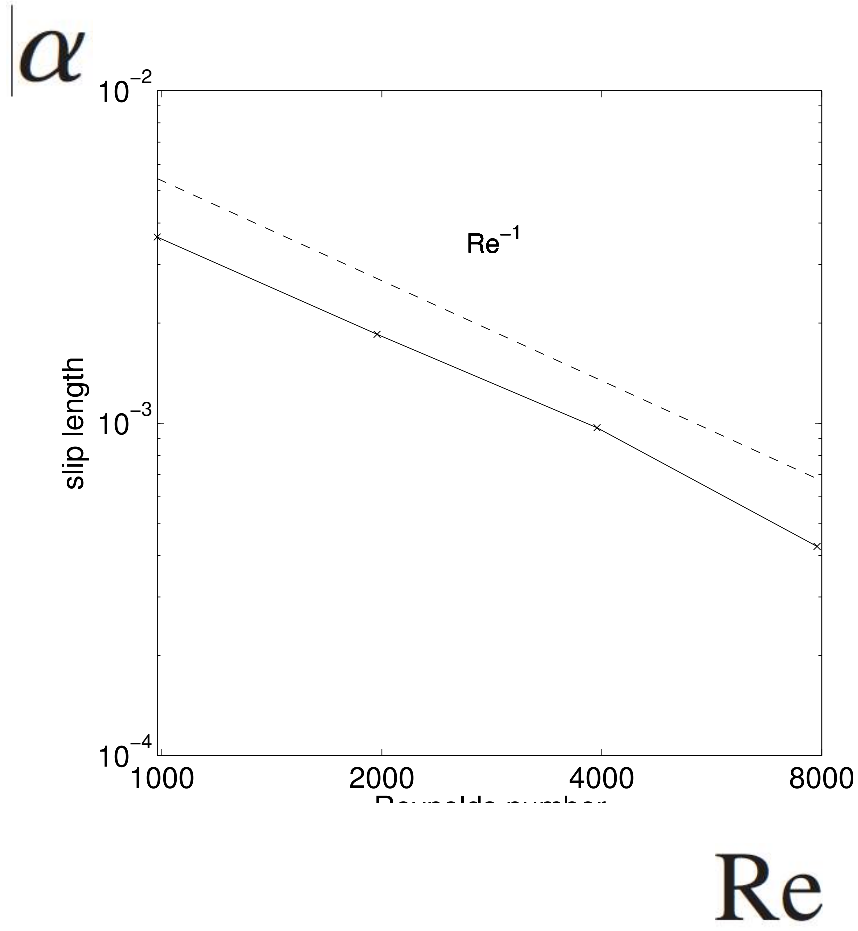
Navier boundary conditions

The wall-normal velocity u_1
is negligible,

The wall-parallel velocity u_2 is much larger and such that

$$u_2 + \alpha(\mathrm{Re}, \eta, N)\partial_1 u_2 \simeq 0$$

This correspond to Navier boundary conditions with a slip length α

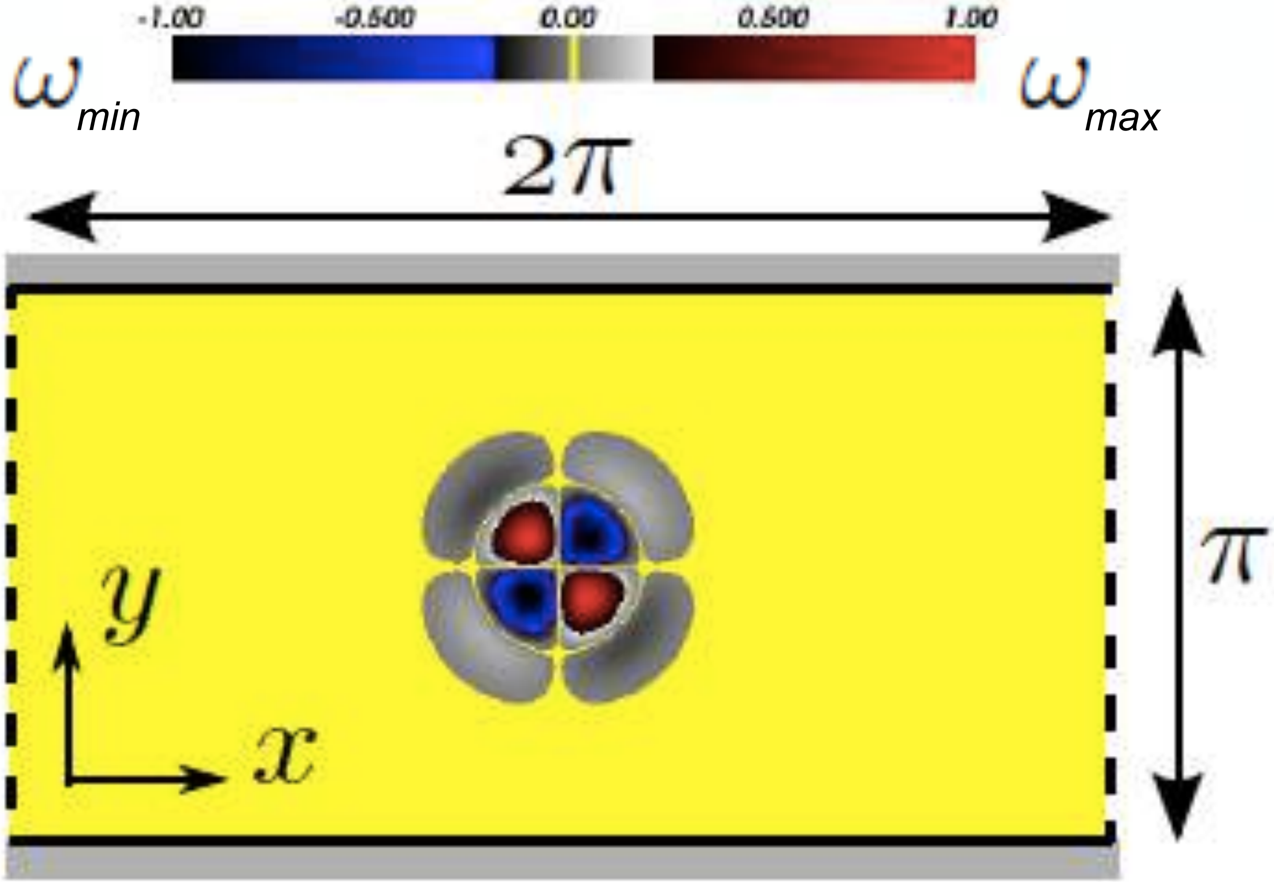


*R. Nguyen van yen, M. F. and K. Schneider, PRL, **106**(18), 2011*

Comparison Navier-Stokes and Euler-Prandtl

Initial vorticity field: vortex quadrupole

No-slip wall



No-slip wall

$$\psi_i(x, y) = Axy \exp\left(-\frac{(x - x_0)^2 + (y - y_0)^2}{2s^2}\right)$$

Prandtl equation coupled to Euler

Ansatz for the vorticity field as Re $\to \infty$:

$$\omega(x,y) = \omega_E(x,y) + \nu^{-1/2}\omega_P(x,\nu^{-1/2}y) + \omega_R(x,y)$$

Prandtl's variable : $y_P = y / \nu^{1/2}$

$$\partial_t \omega_P + \boldsymbol{\nabla}.(\mathbf{u}_P \omega_P) = \partial^2_{y_P}\omega_P$$

$$\omega_P(x, y_P, 0) = 0$$

$$\psi_P(x, y_P, t) = \int_0^{y_P} \mathrm{d}y'_P \int_0^{y'_P} \mathrm{d}y''_P \omega_P(x, y''_P, t)$$

$$\partial_{y_P}\omega_P(x, 0, t) = -\partial_x p_E(x, 0, t),$$

where p_E is the pressure calculated from ω_E which is the vorticity given by Euler equation

Prandtl solver

- Artificial boundary condition: $\partial_{y_P}\omega_P = 0$ at $y_P = 64$
- Spatial discretization: Fourier in x
  and compact finite differences of 5th order in y
- Time discretization: low storage third order Runge-Kutta in t

- Neumann boundary condition for vorticity:

  $$\partial_{y_P}\omega_P = -\partial_x p_E \text{ at } y_P = 0$$

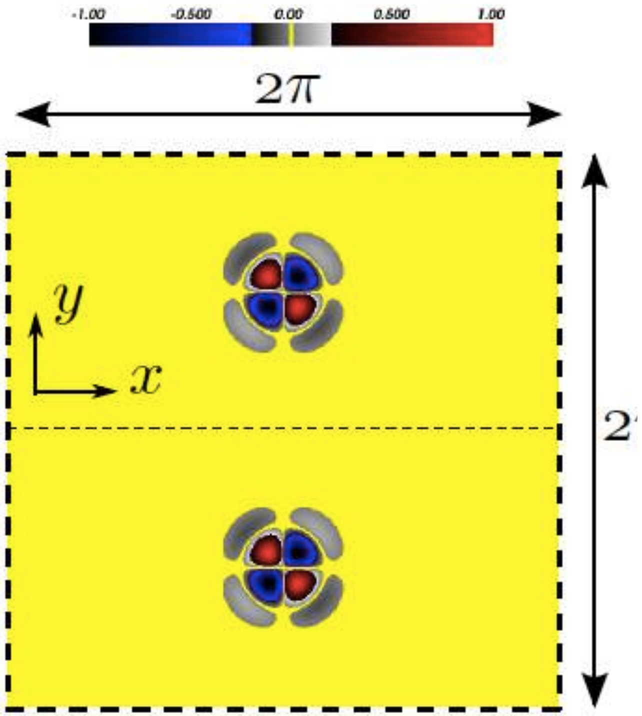
  where p_E is the pressure calculated from ω_E

- To close the system we impose

  $$\partial^2_{y_P}\omega_P = 0 \text{ at } y_P = 64$$

  which is consistent with the rapid decay of ω_P

Euler solver

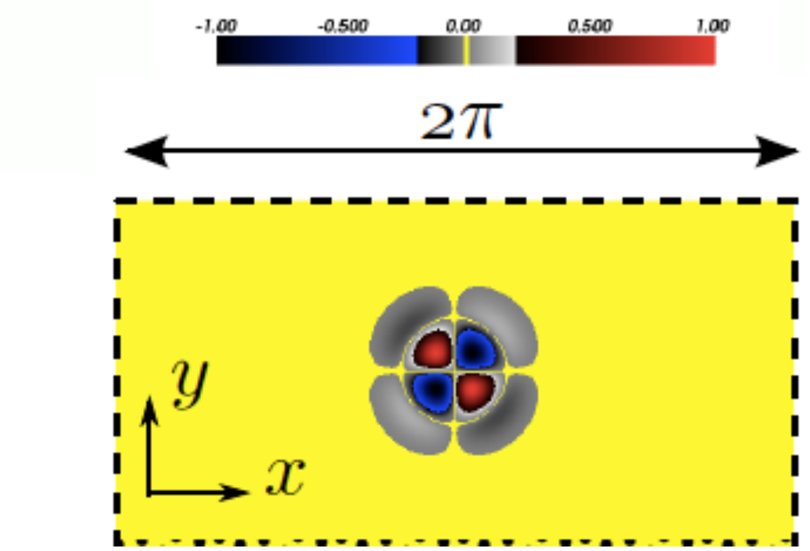


- Use mirror symmetry around $y = 0$ to impose boundary condition.
- Spatial discretization: Fourier pseudo-spectral with hyperdissipation, $k_{max} = 682$.
- Time discretization: third order low storage Runge-Kutta, with exponential propagator for the viscous term.

Navier-Stokes solver

Fourier/compact finite differences (5th order)

- Similar to the one for the Prandtl equations, except that non-uniform grids are used in the *y* direction.
- Two linear integral constraints are applied on vorticity to satisfy the no-slip boundary conditions in *y*.
-Integrating factor for the viscous term and 3rd order Runge-Kutta
-for the advection term.



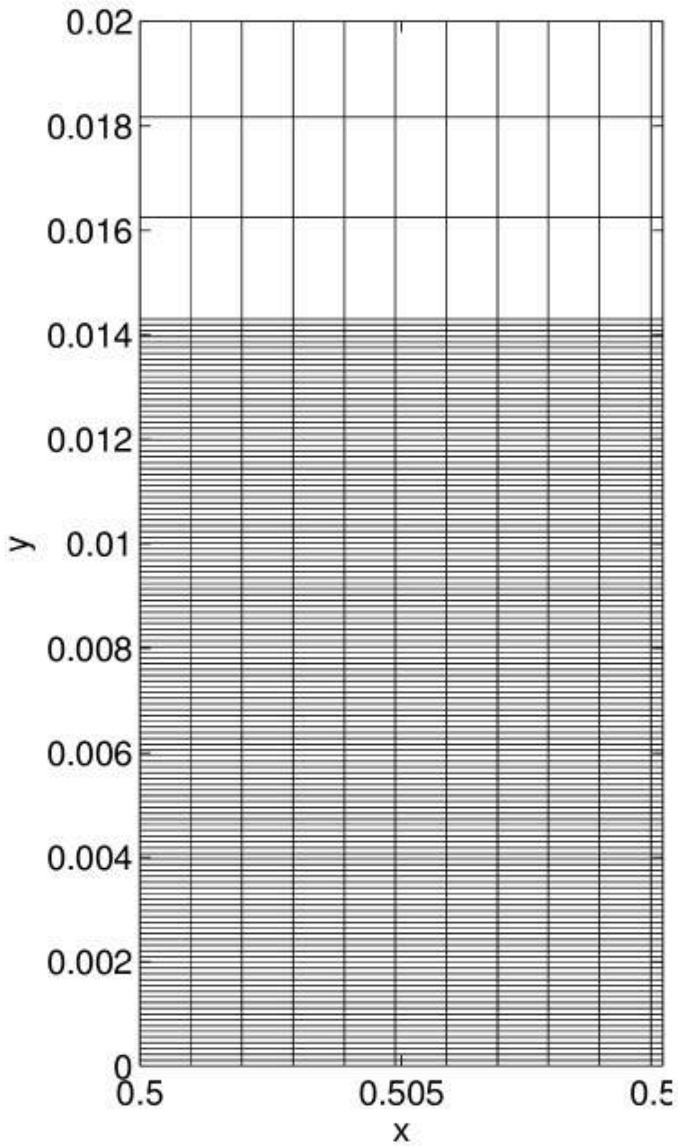
$N_x = 1024$

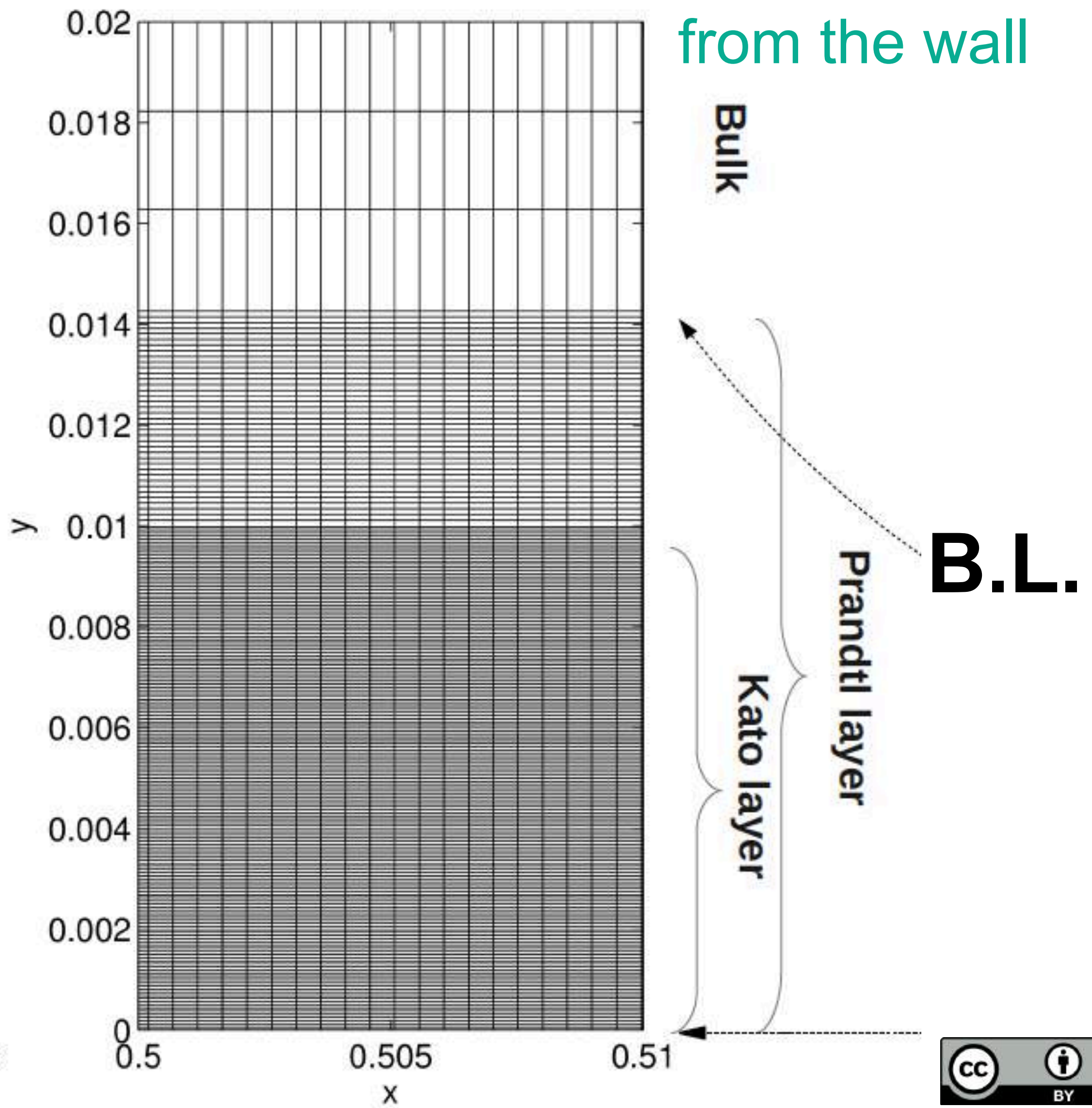
$N_y = 385 - 449$

Computational grid

Before t=54

After t=54 when B.L. detachs from the wall





Comparison Navier-Stokes and Euler-Prandtl

Navier-Stokes solver

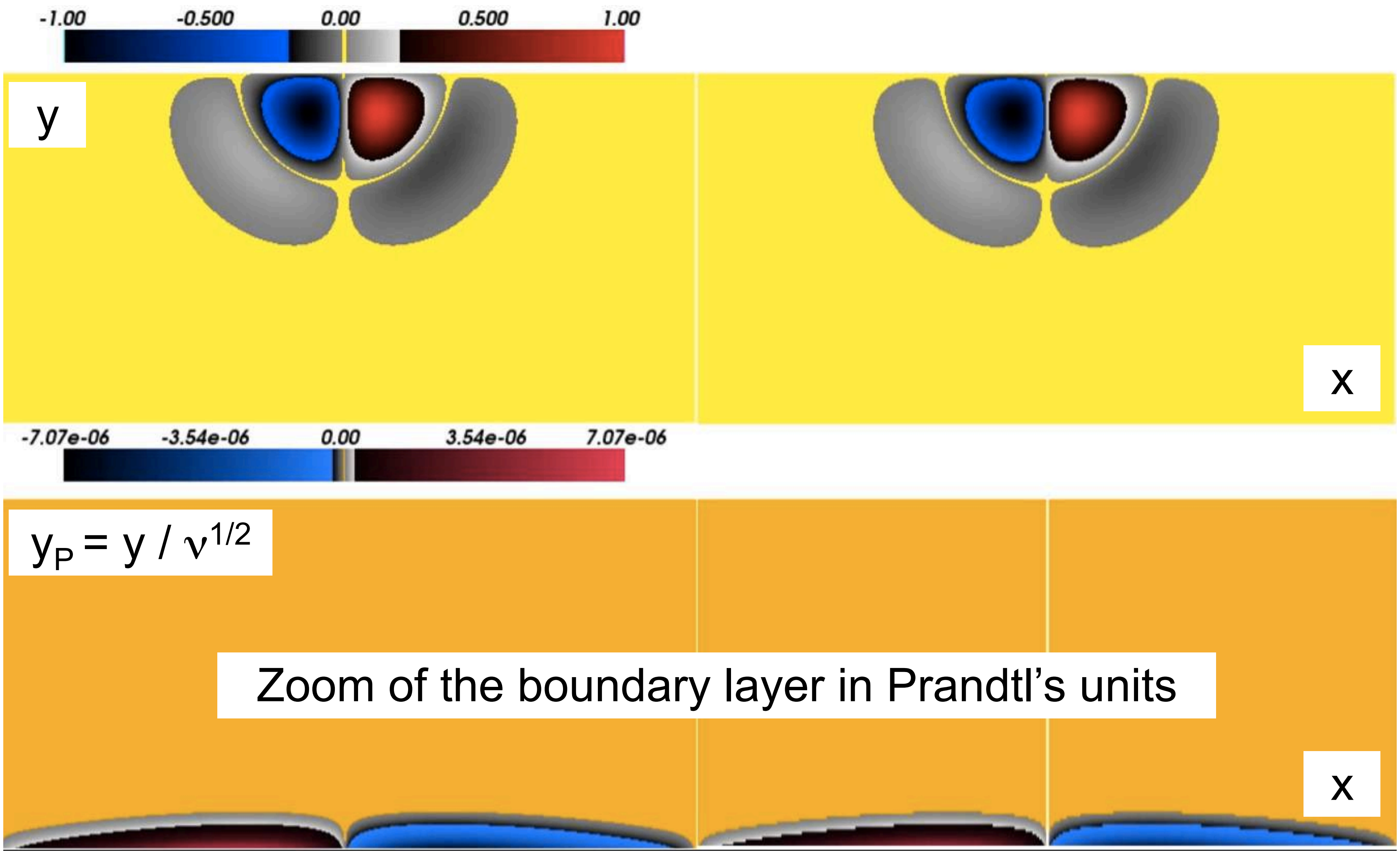
- Fourier in x and compact finite differences of 5th order with non-uniform grid in y.
- Third order Runge-Kutta in t.
- Periodic in x and no-slip boundary conditions in y.

Euler solver

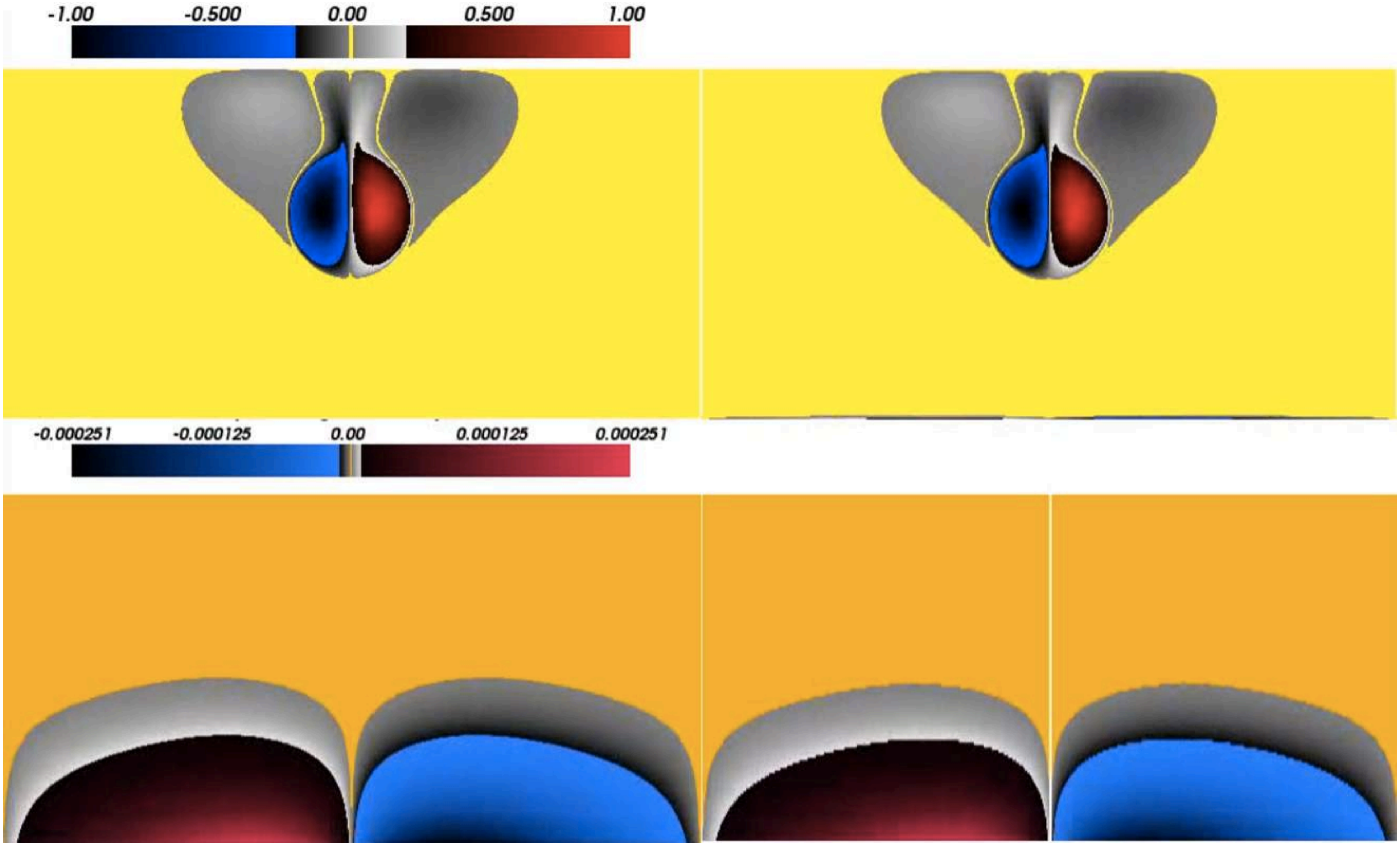
- Fourier with hyperdissipation in x and y.
- Third order Runge-Kutta in t.
- Mirror-symmetry around y=0 to impose boundary conditions.

Prandtl solver

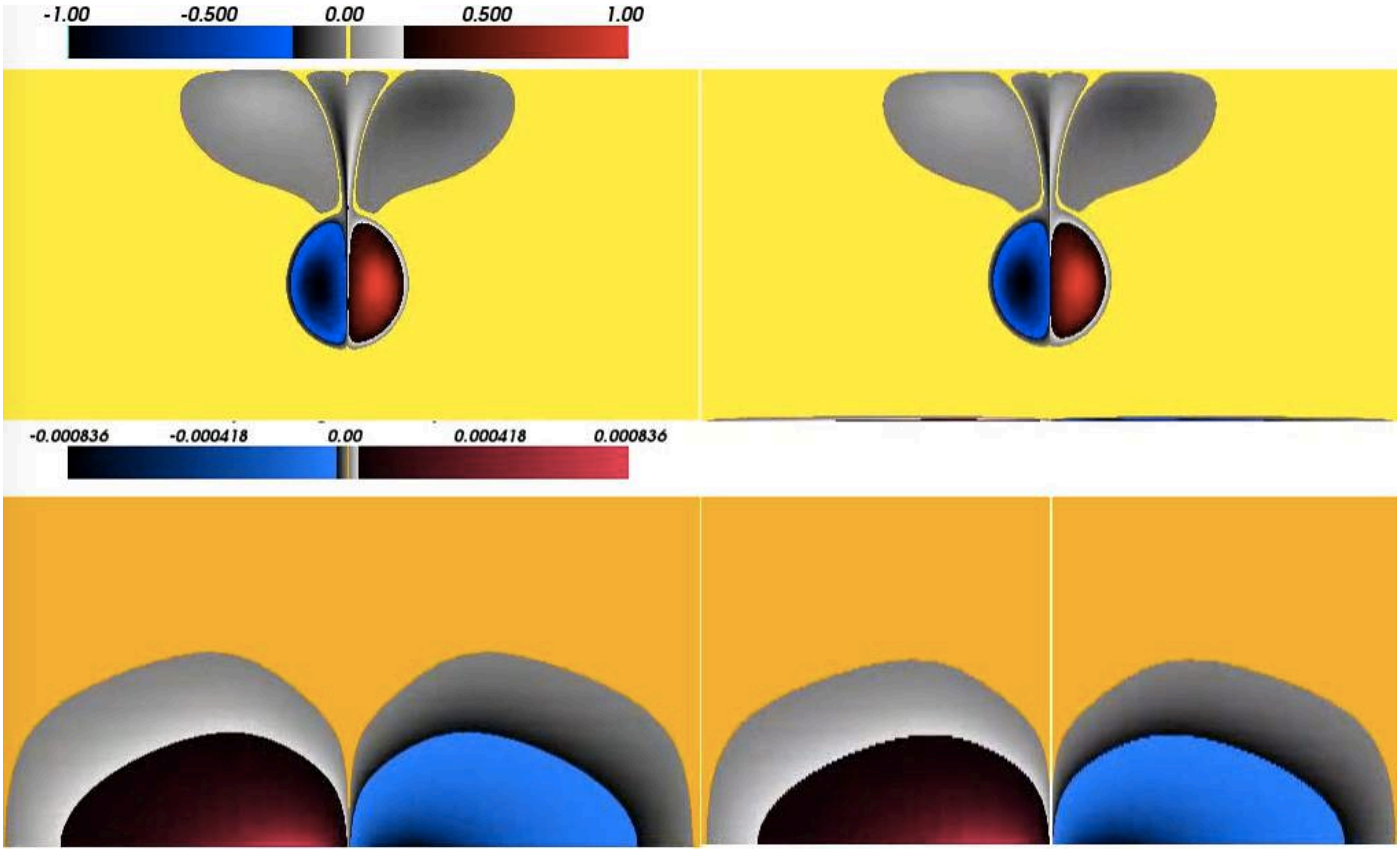
- Second order finite differences in x and y.
- Second order semi-implicit Runge-Kutta in t.
- Neumann boundary condition at y=0 when inverting.



Euler and Prandtl **Navier-Stokes**

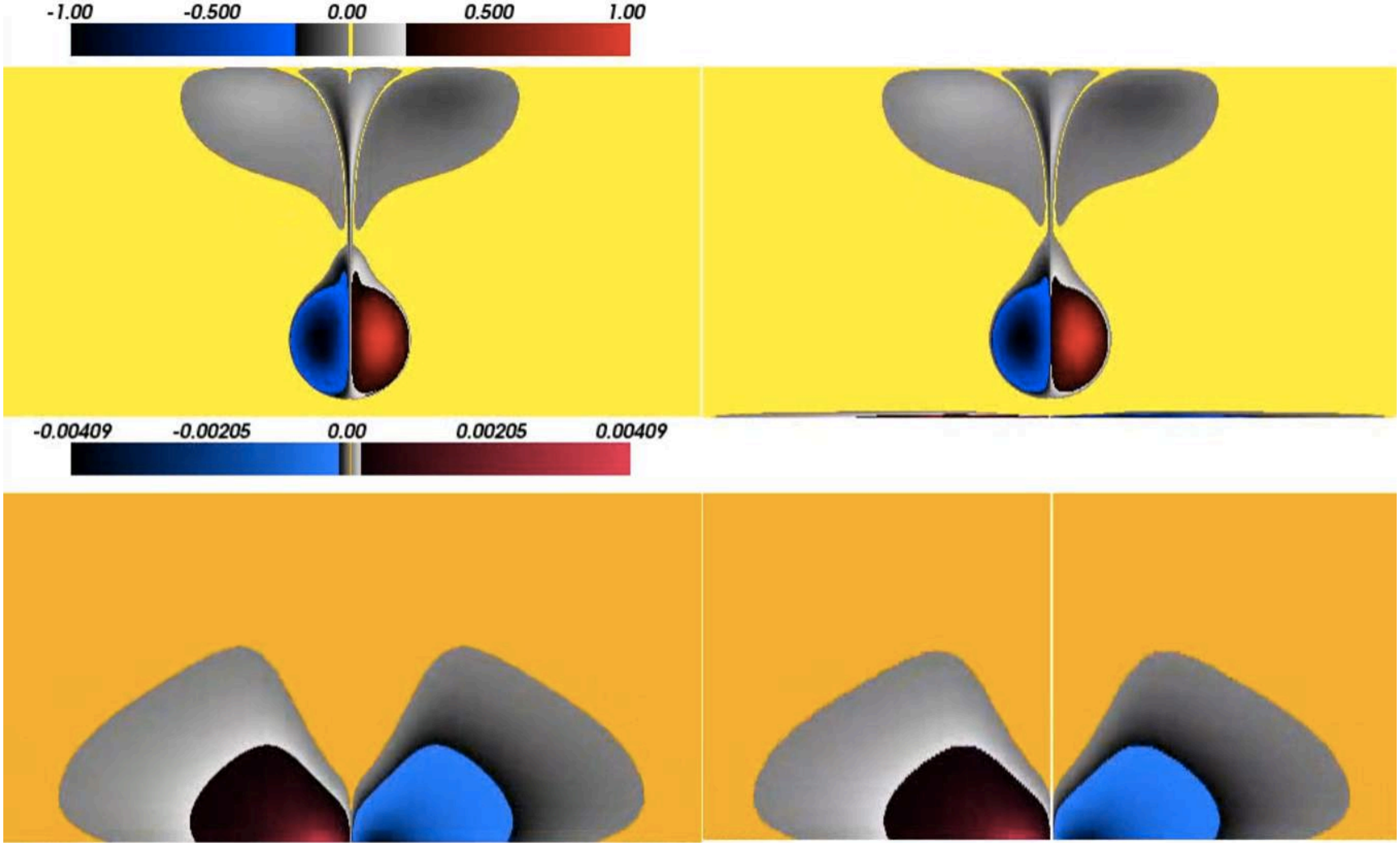


Euler and Prandtl **Navier-Stokes**

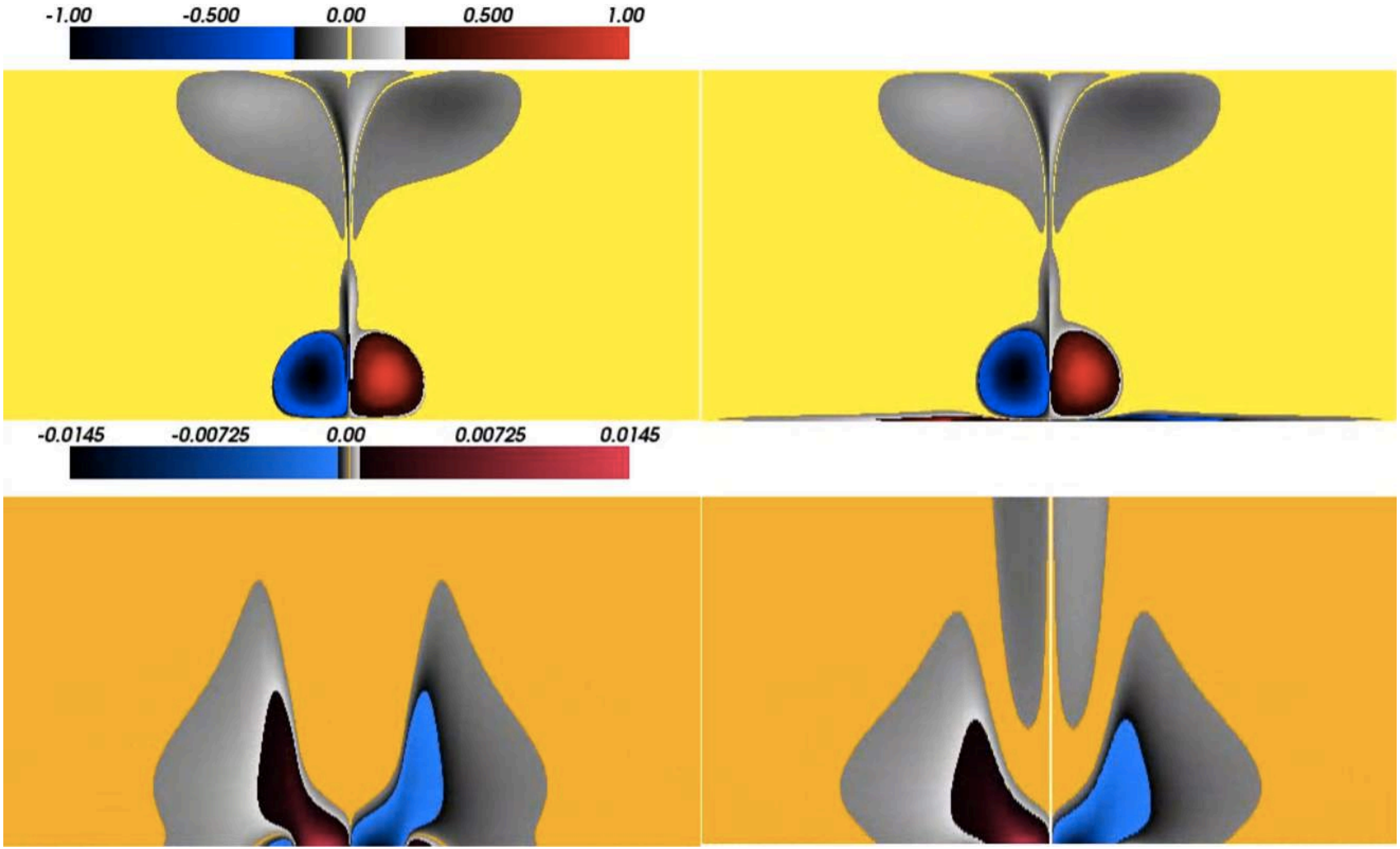


Euler and Prandtl

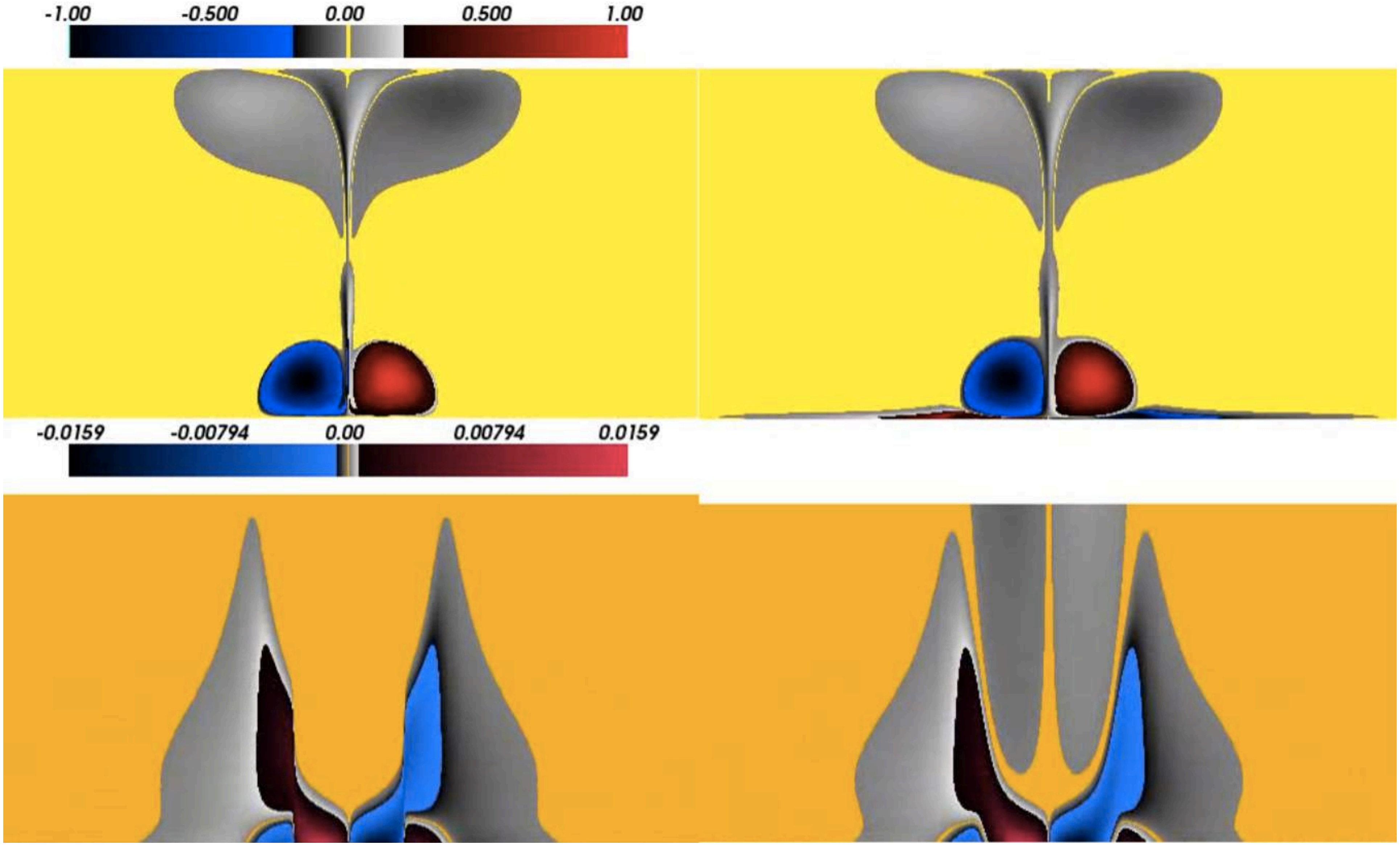
Navier-Stokes



Euler and Prandtl **Navier-Stokes**

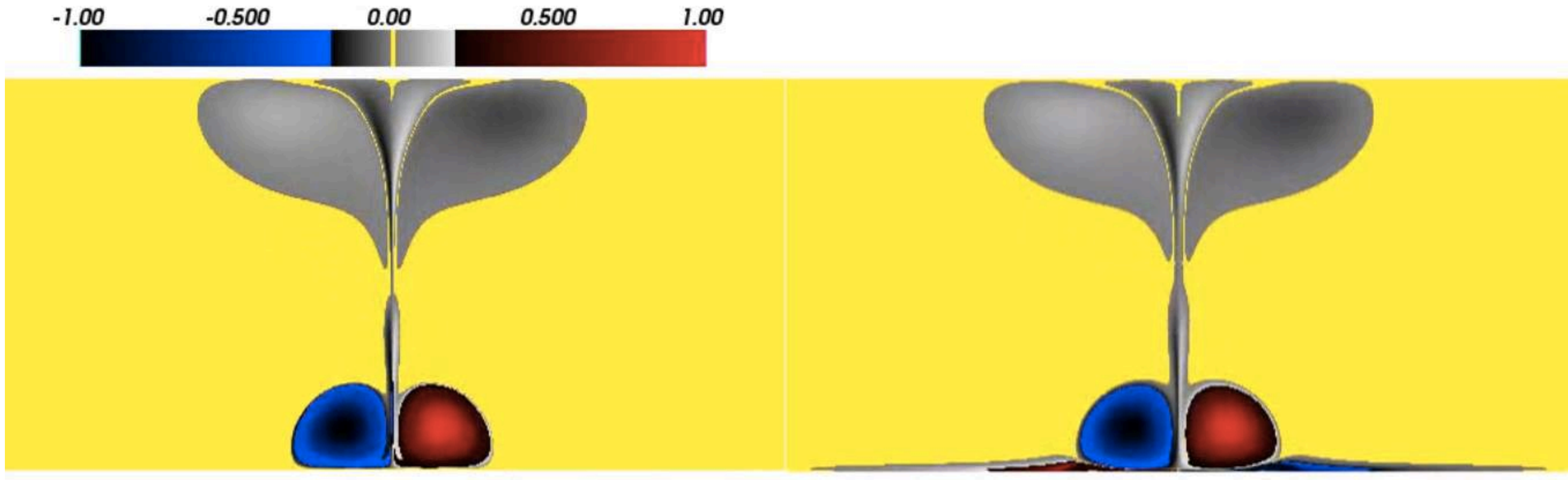


Euler and Prandtl **Navier-Stokes**

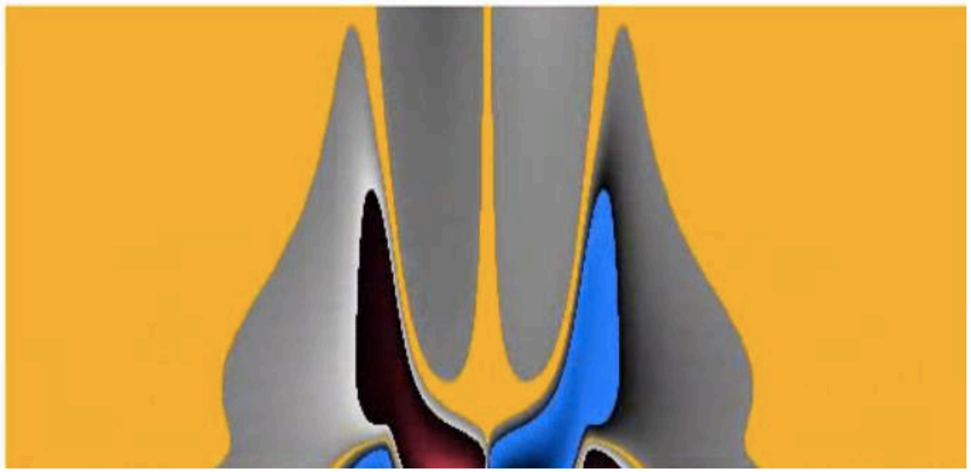


Euler and Prandtl

Navier-Stokes

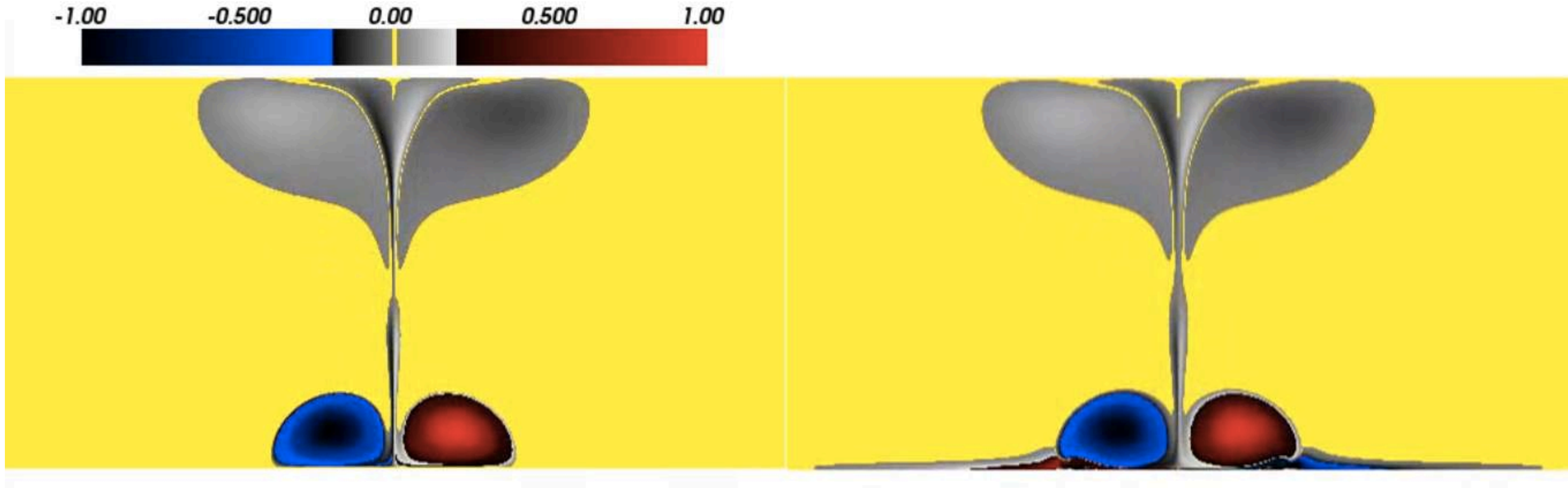


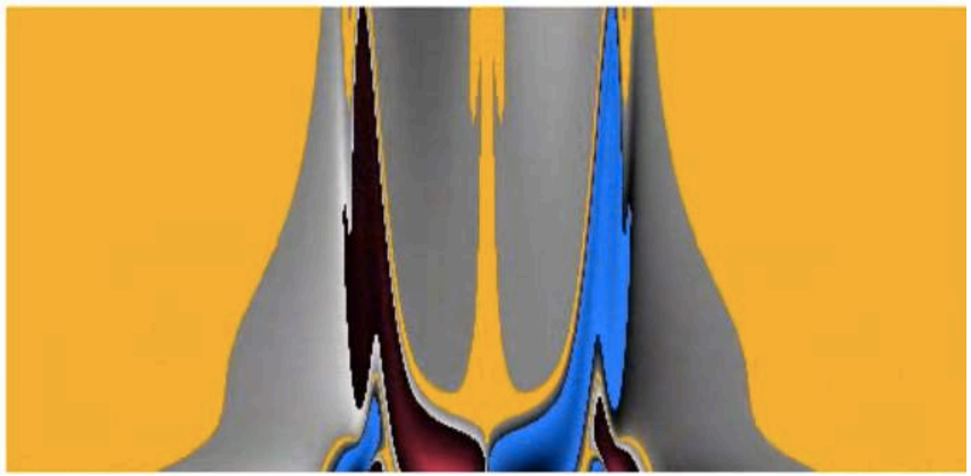
Prandtl's solution
no more exists
after t= 55.8



Euler

Navier-Stokes

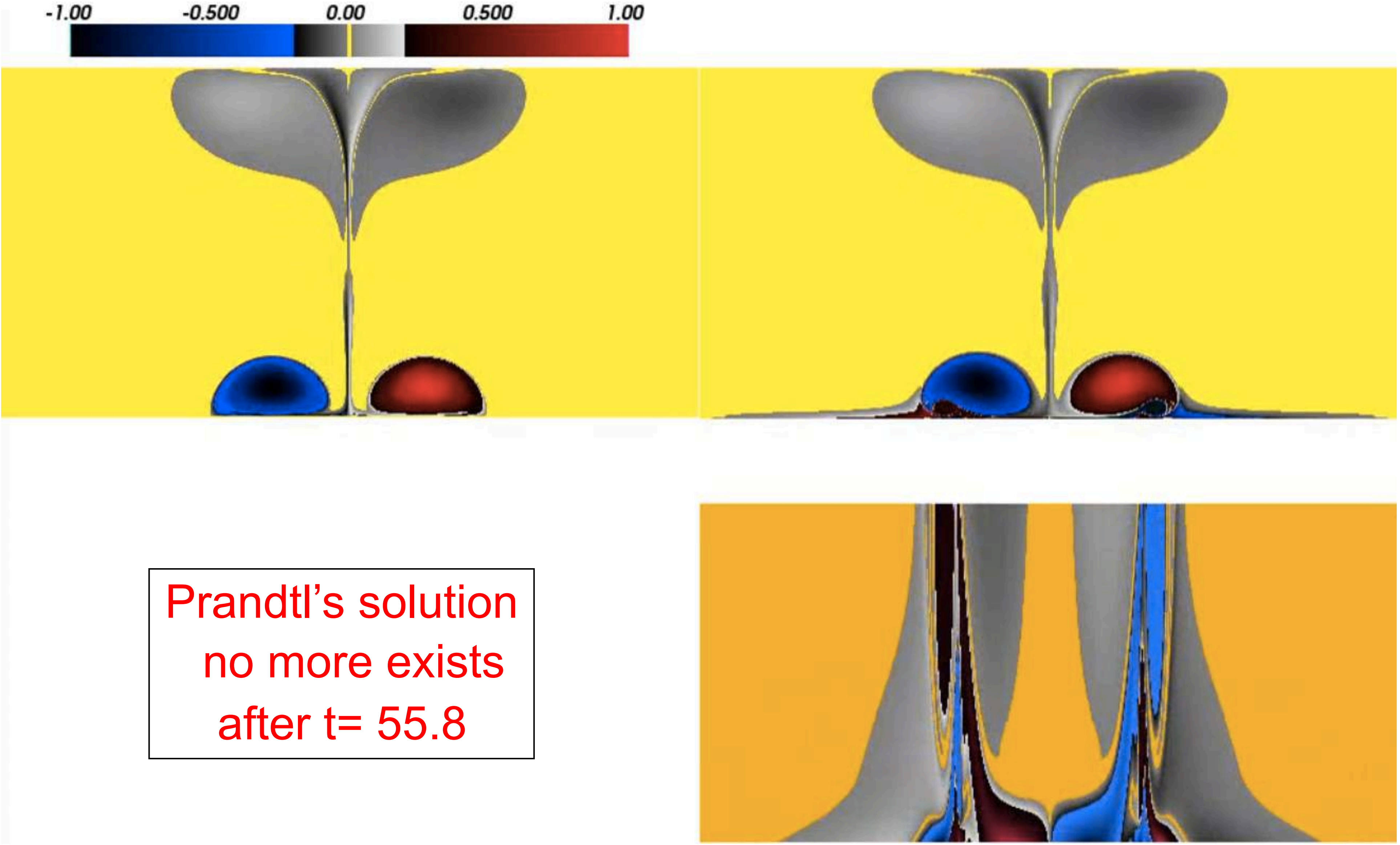




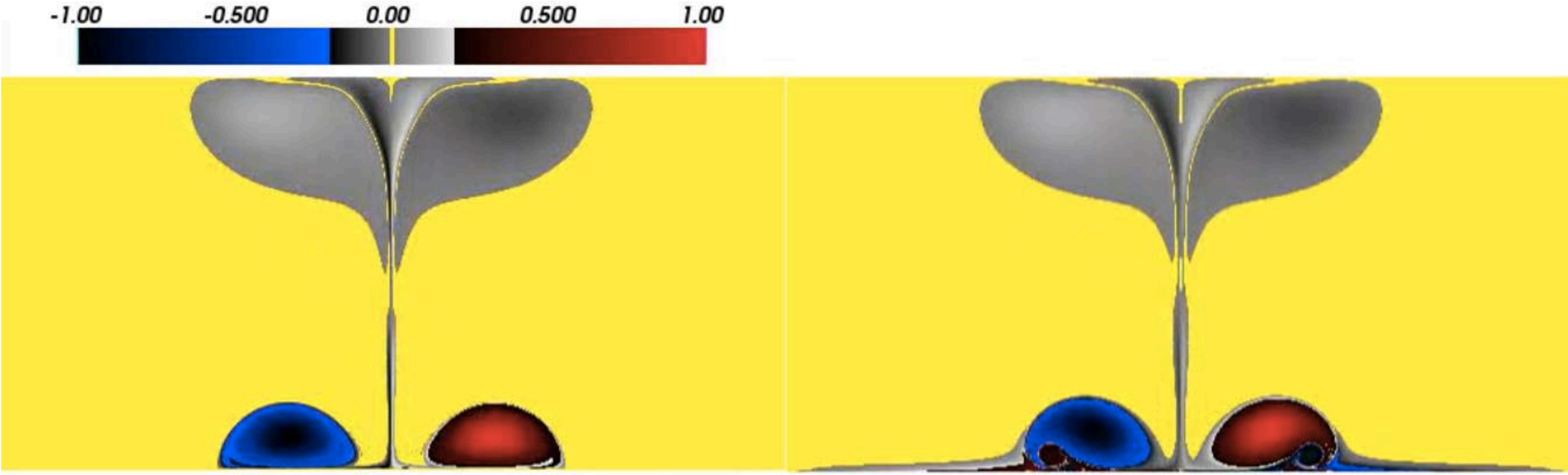
Prandtl's solution
no more exists
after t= 55.8

Euler

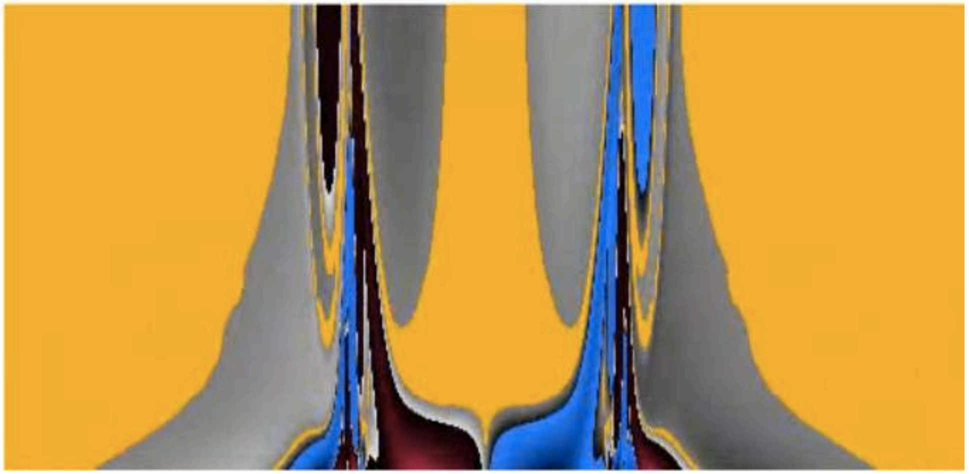
Navier-Stokes



Euler **Navier-Stokes**

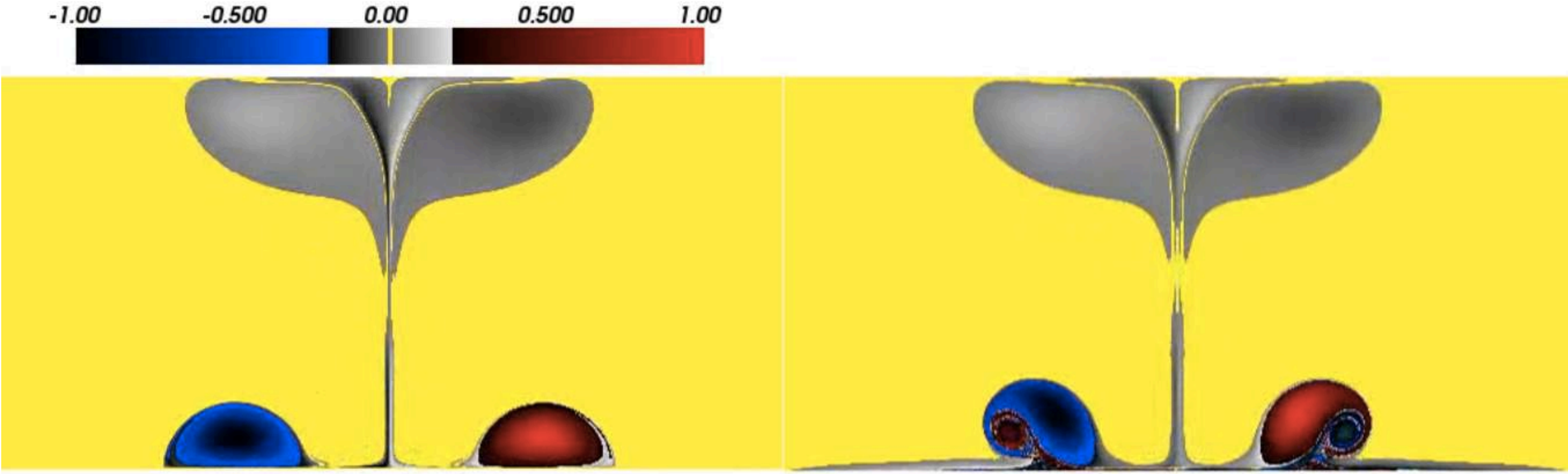


Prandtl's solution no more exists after t= 55.8

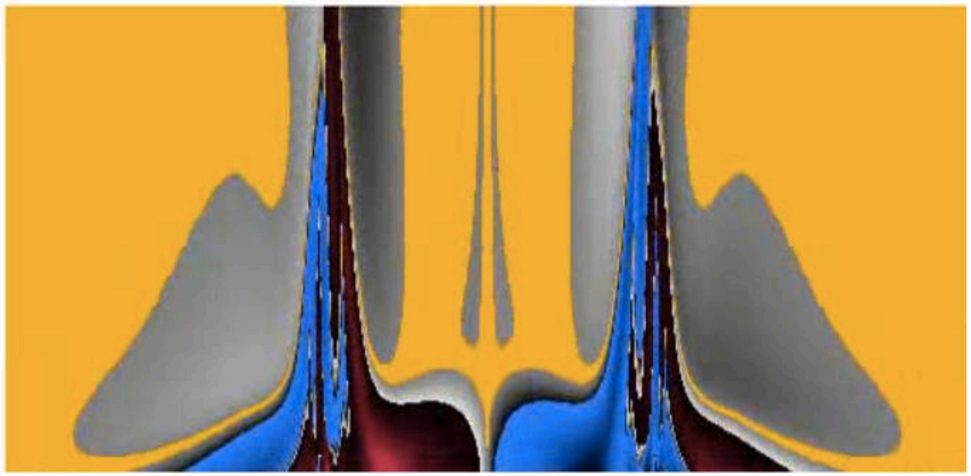


Euler

Navier-Stokes

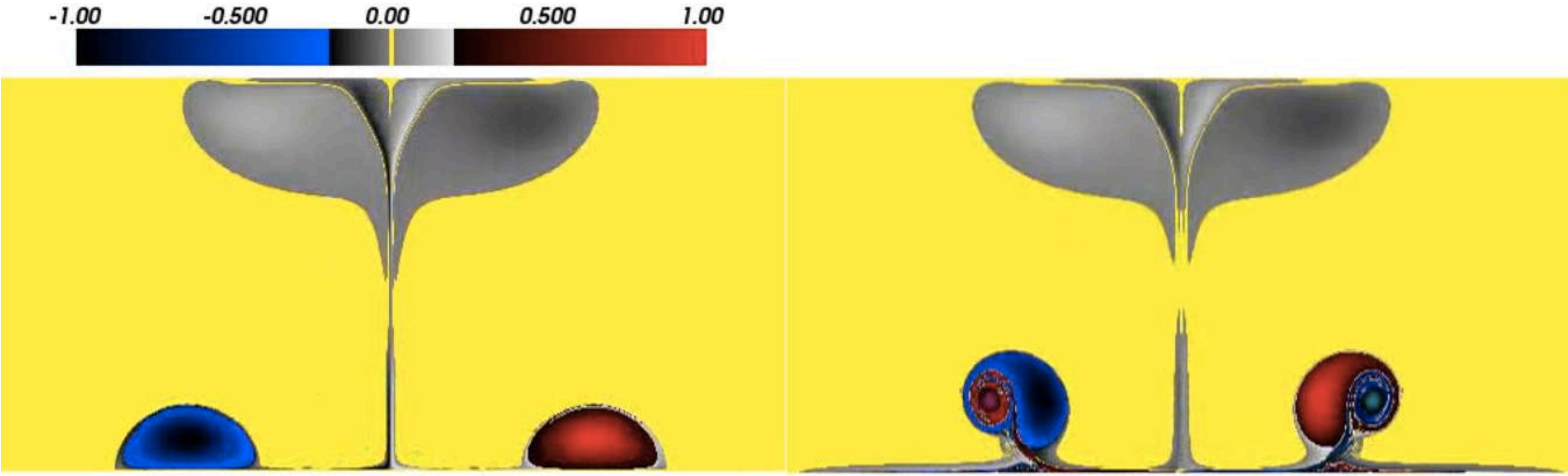


Prandtl's solution
no more exists
after t= 55.8

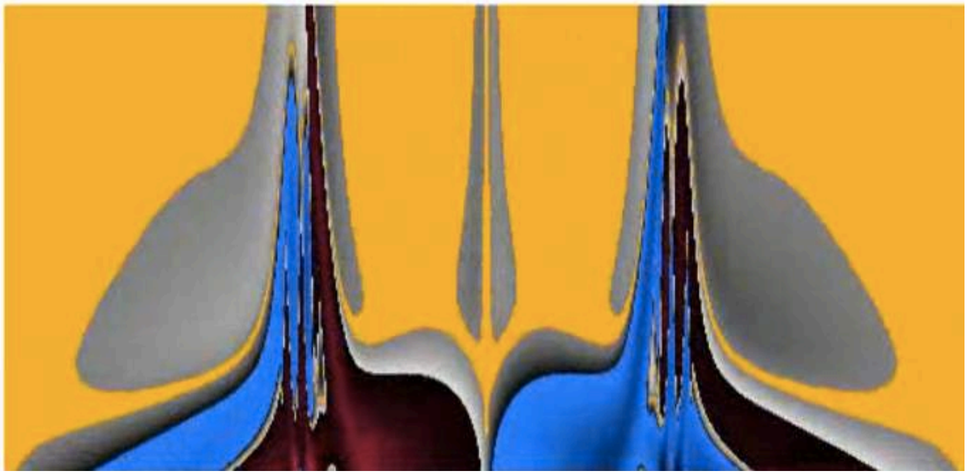


Euler

Navier-Stokes

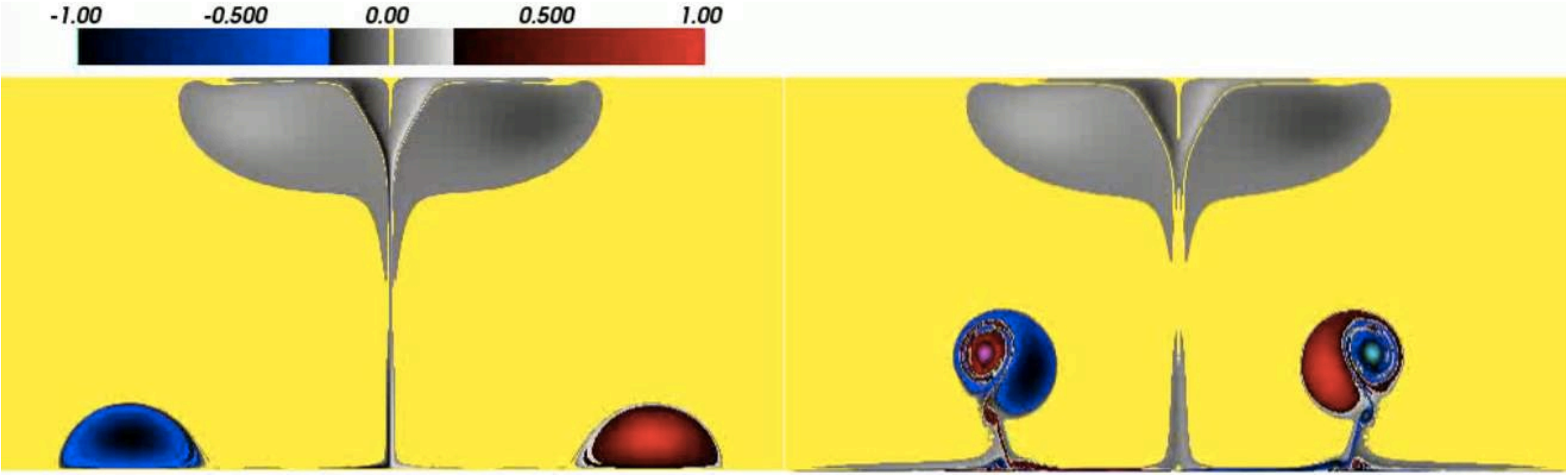


Prandtl's solution
no more exists
after t= 55.8

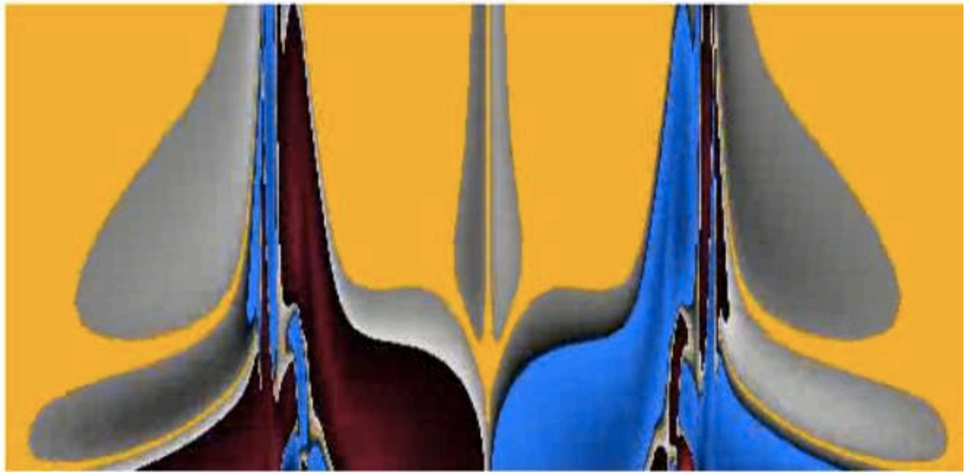


Euler

Navier-Stokes

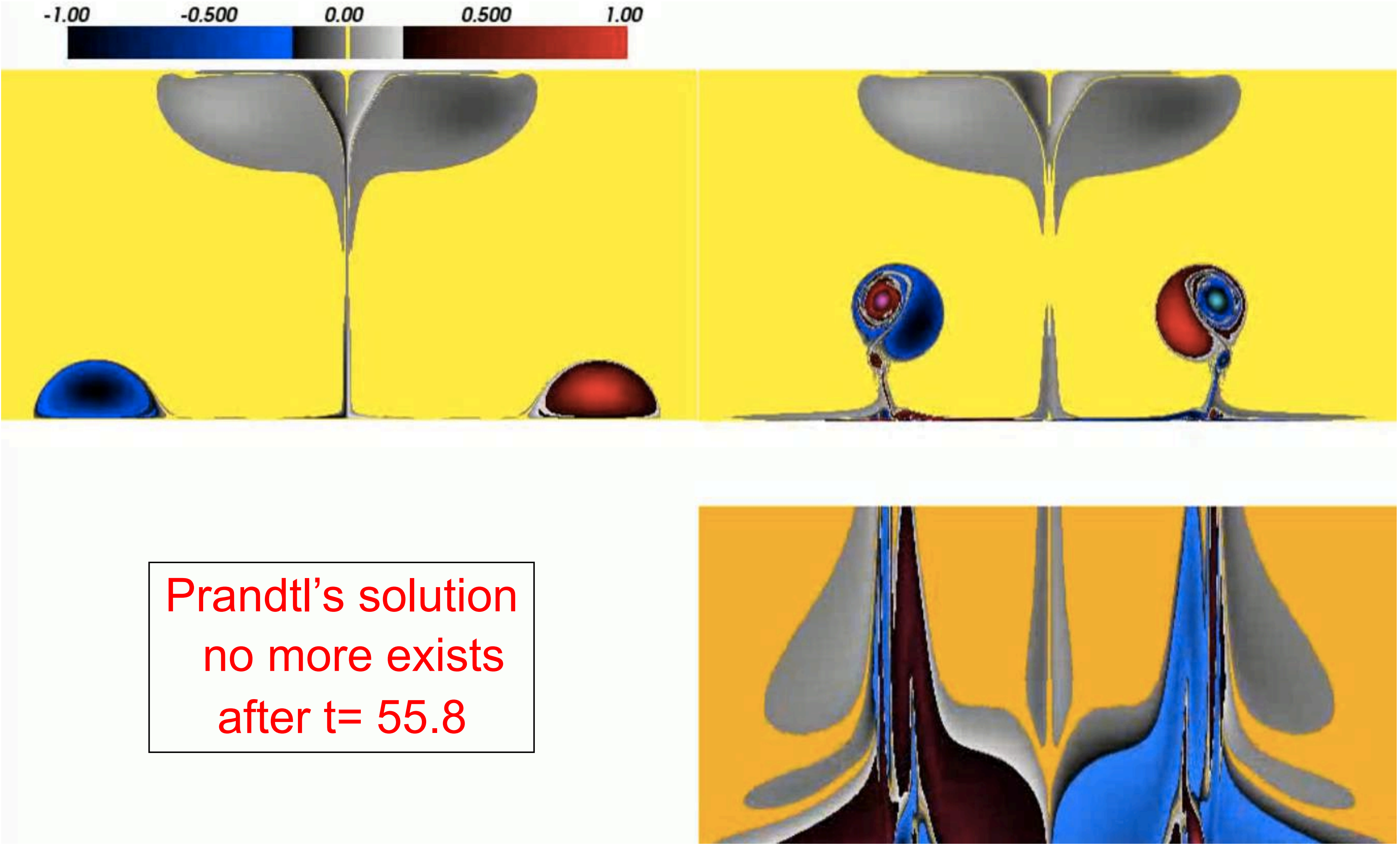


Prandtl's solution
no more exists
after t= 55.8

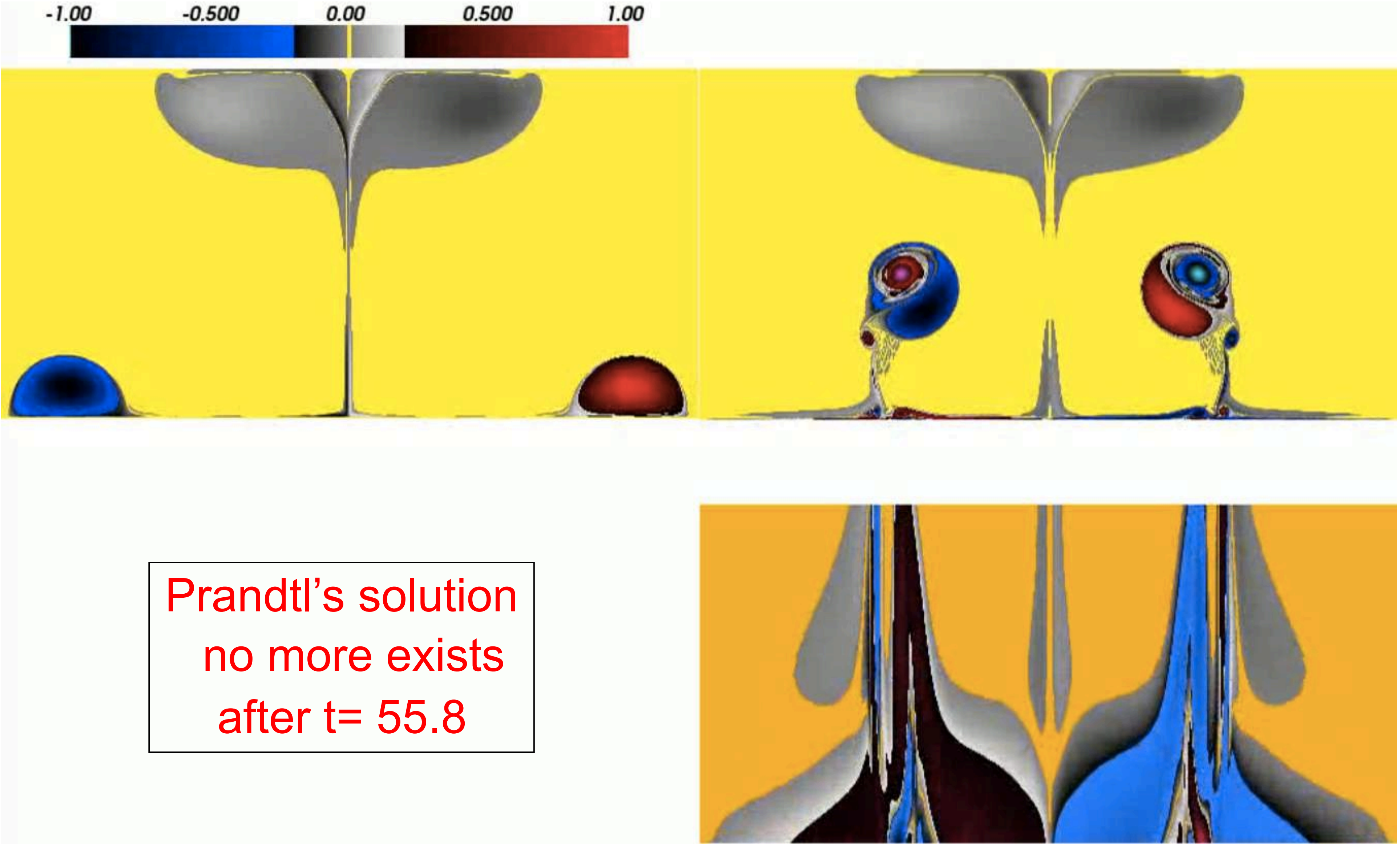


Euler

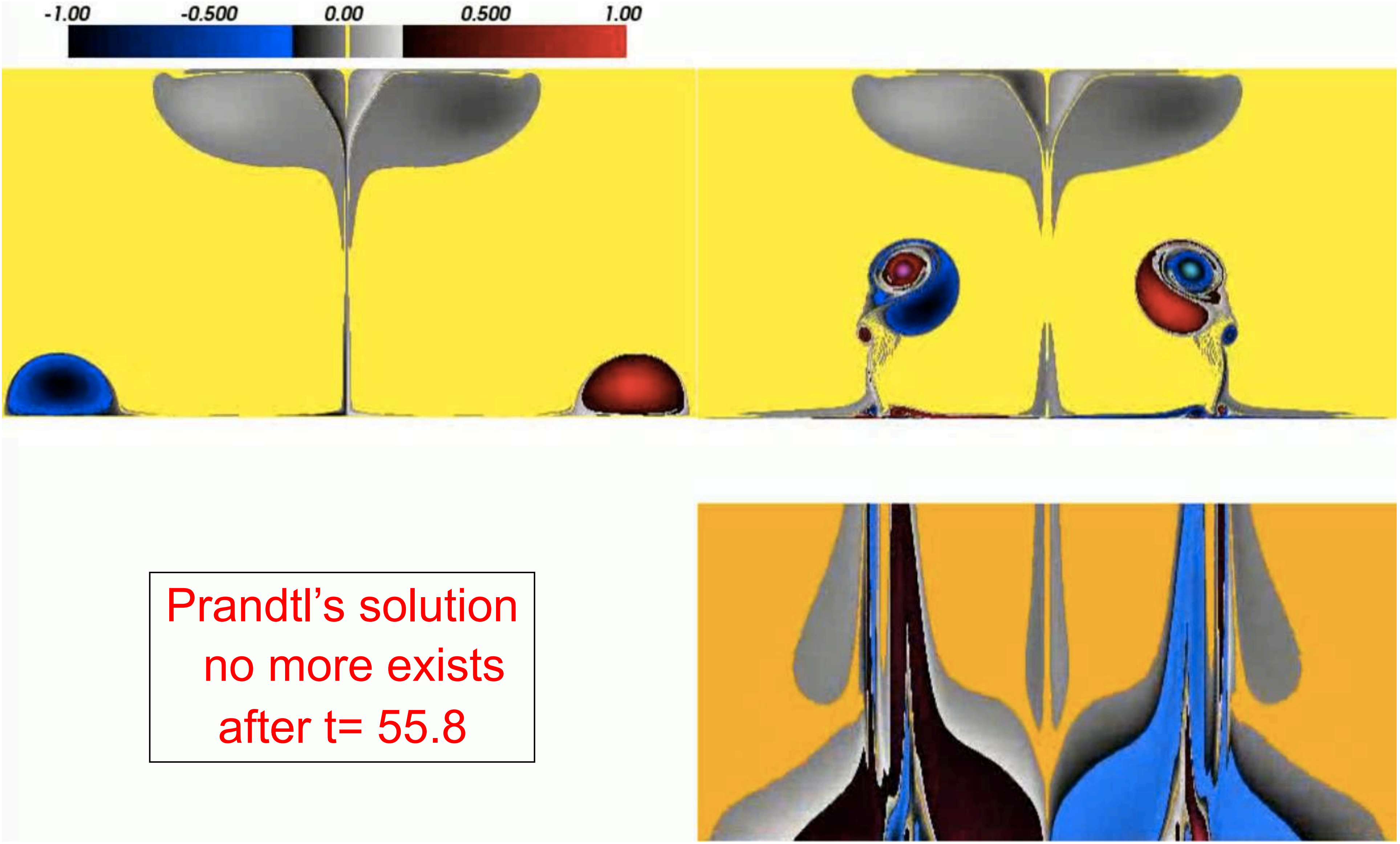
Navier-Stokes



Euler **Navier-Stokes**



Euler **Navier-Stokes**



Prandtl's solution no more exists after t= 55.8

Euler

Navier-Stokes

Prandtl's singularity

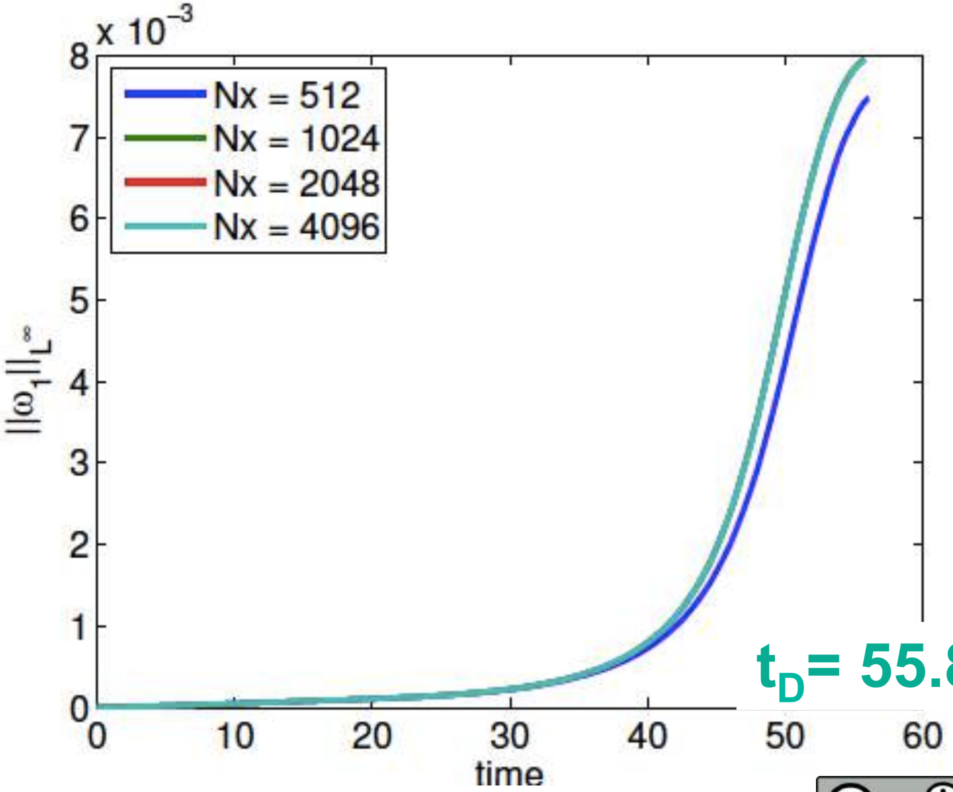
Prandtl equation has well-known finite time singularity

- $|\partial_x \omega_1|$ and $u_{1,y}$ blows up,
- ω_1 remains bounded.

L. L. van Dommelen and S. F. Shen., 1980 J. Comp. Phys., ***38****(2)*

This is observed in our computations as expected,

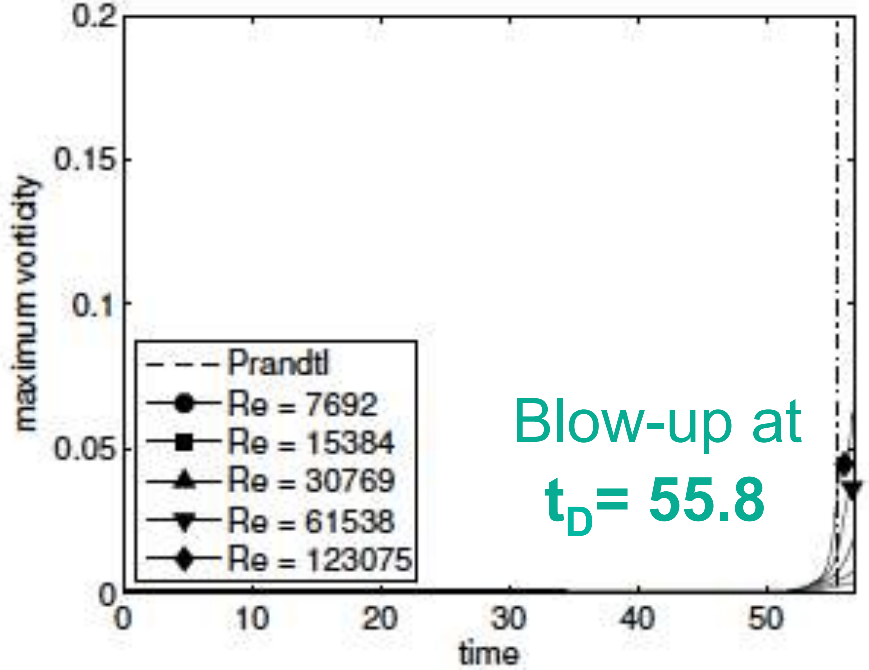
for $t \to t_D \simeq 55.8$

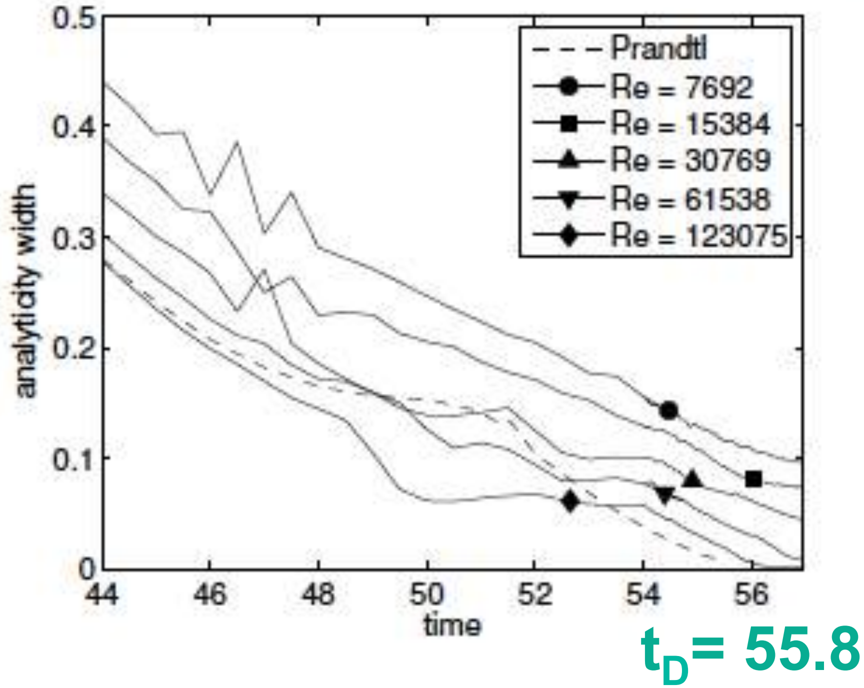


$t_D = 55.8$

Prandtl solution's blow-up at t_D=55.8

According to Kato's theorem, and since ω_1 remains bounded uniformly until t_D, we expect that $\mathbf{u}_\nu \xrightarrow[\nu\to 0]{L^2} \mathbf{u}_0$ uniformly on $[0, t_D]$.

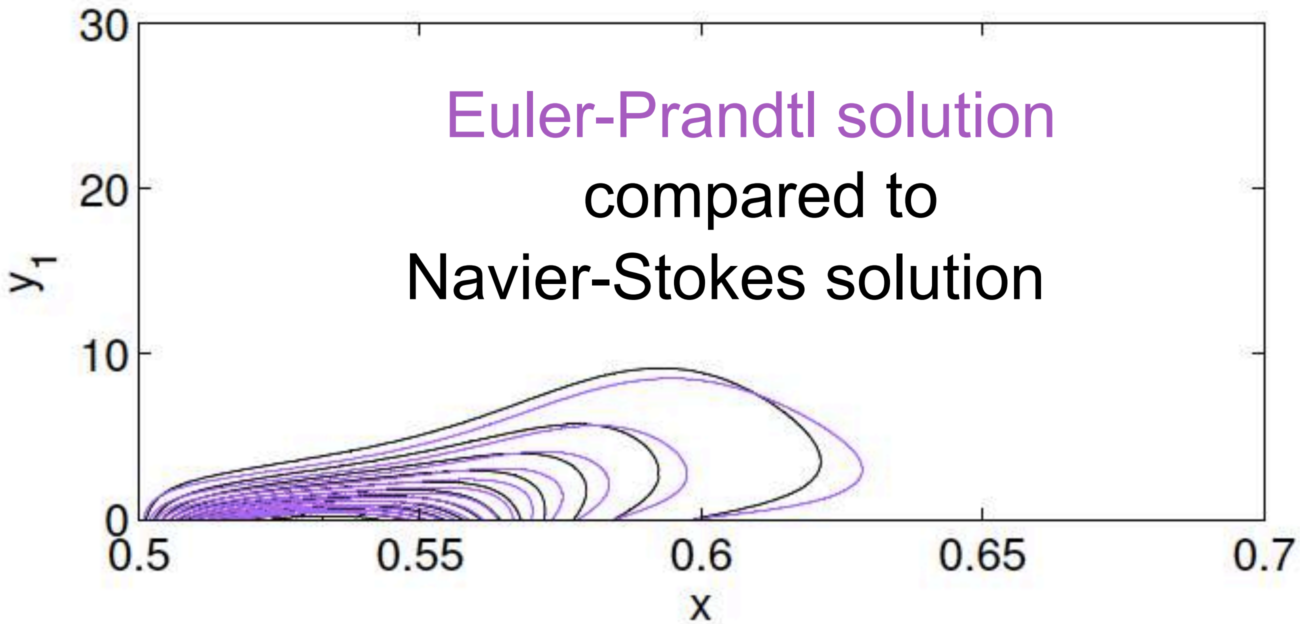


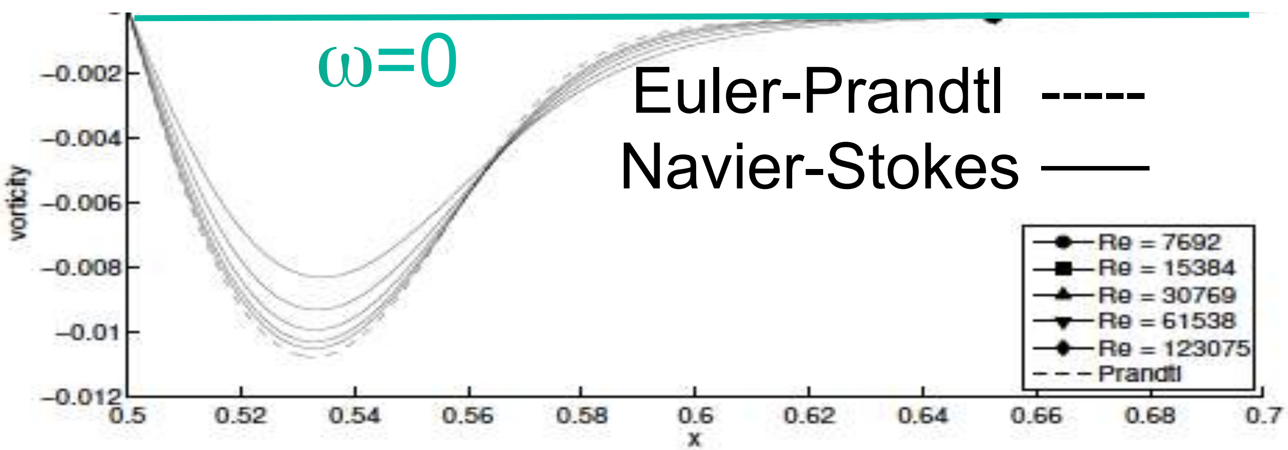


Evolution of vorticity max **Evolution of analyticity strip**

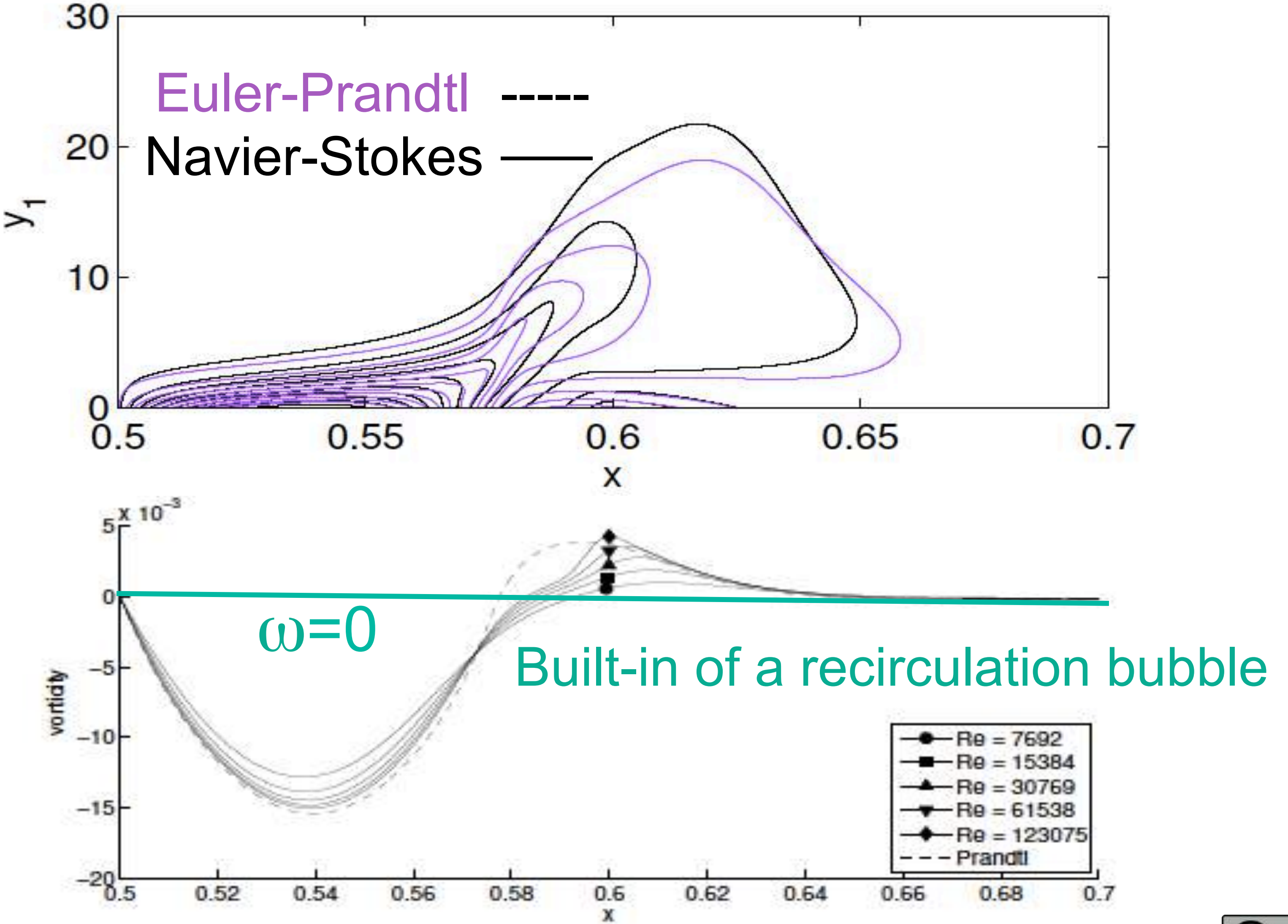
Show convergence!

Vorticity along the wall at t=50 < t_D

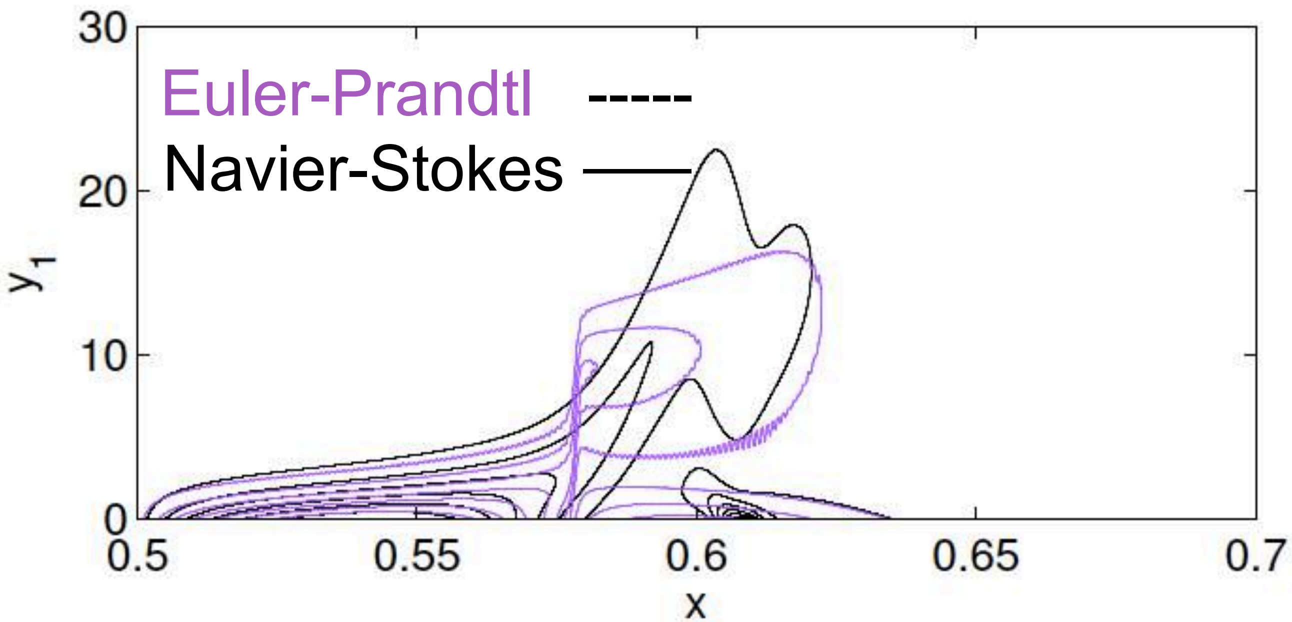


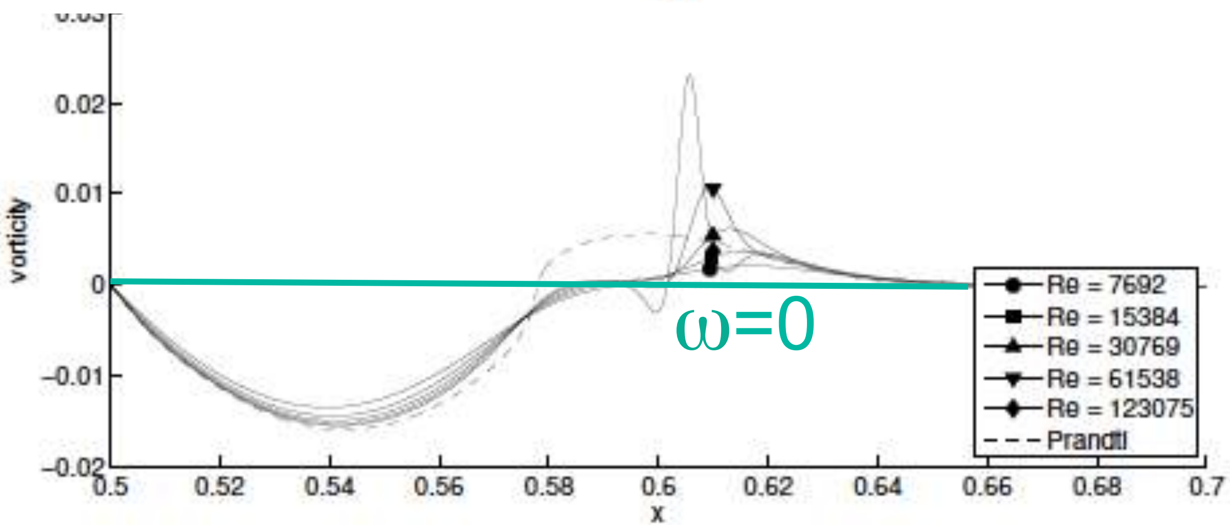


Vorticity along the wall at t=54 < t_D

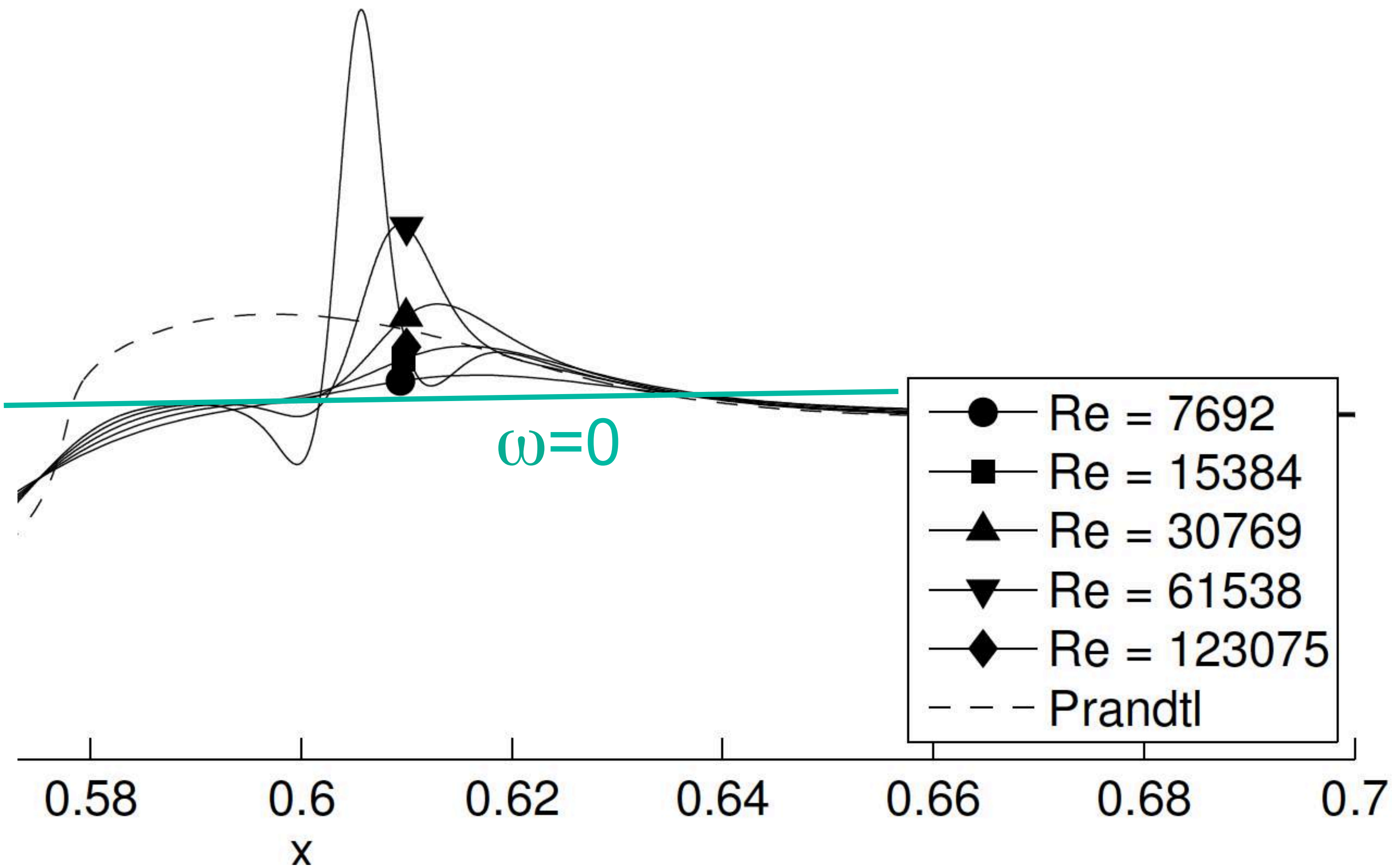


Vorticity along the wall at t=55 < t_D

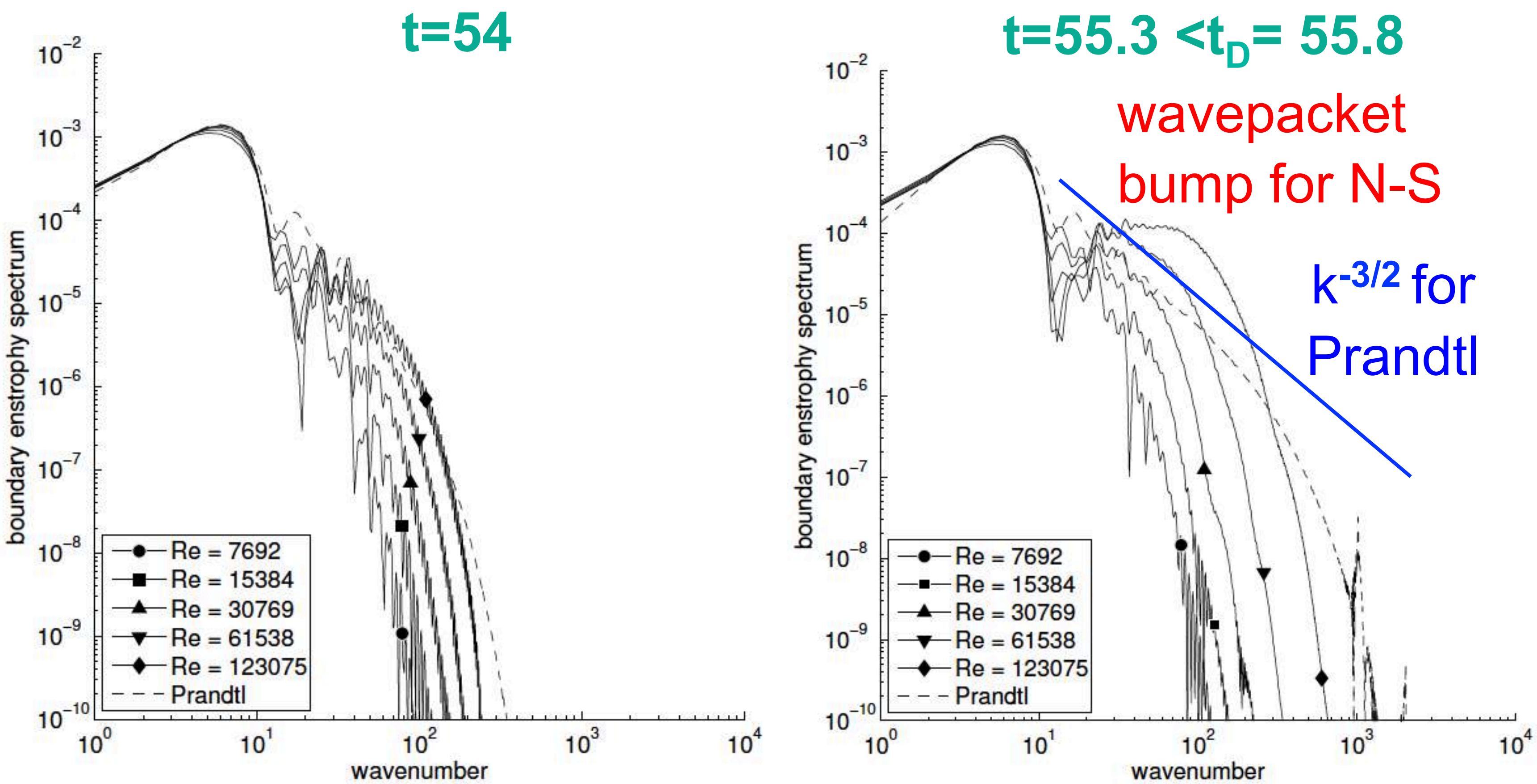




Vorticity along the wall at $t=55.3 < t_D$

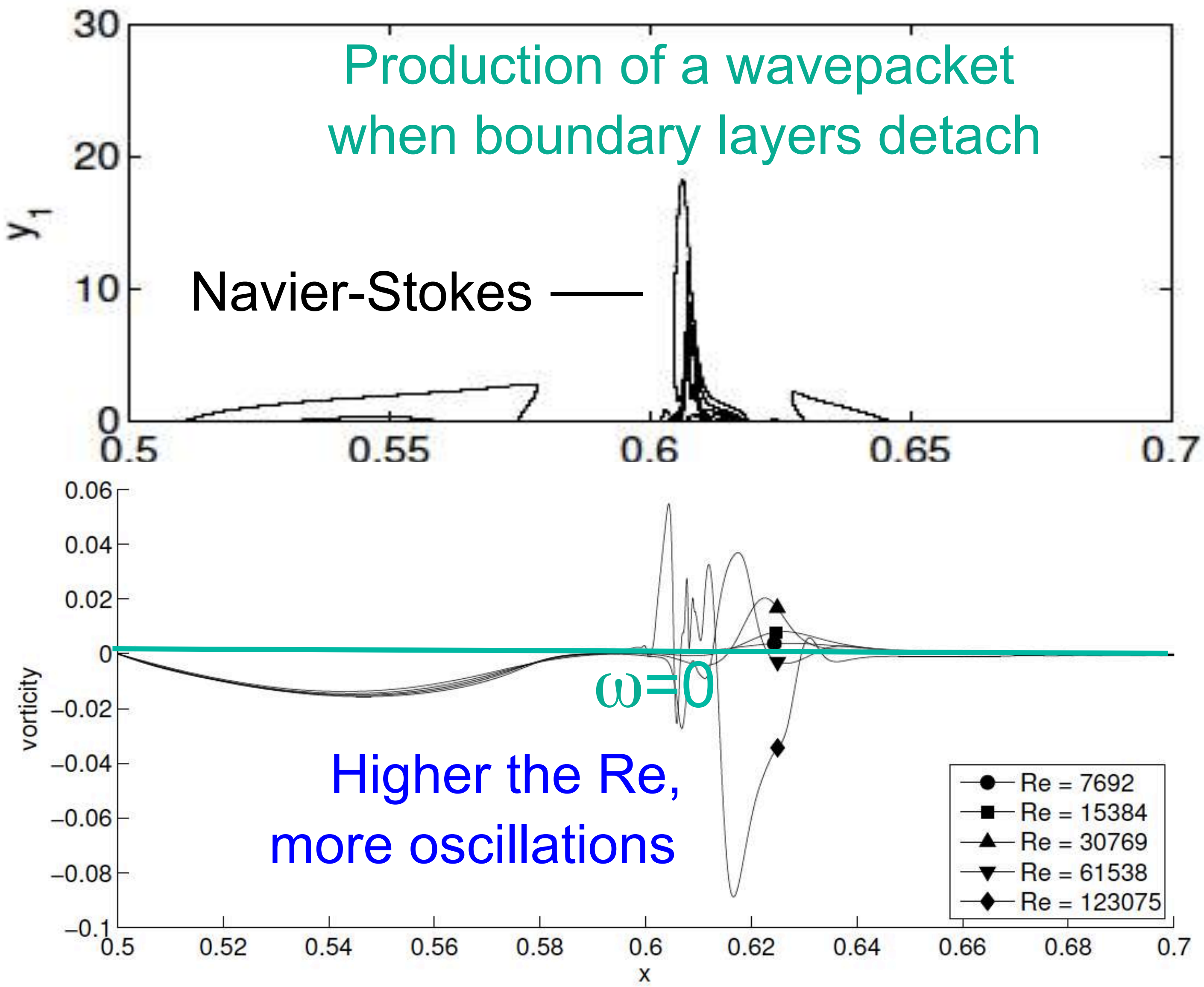


Spectrum of the boundary vorticity

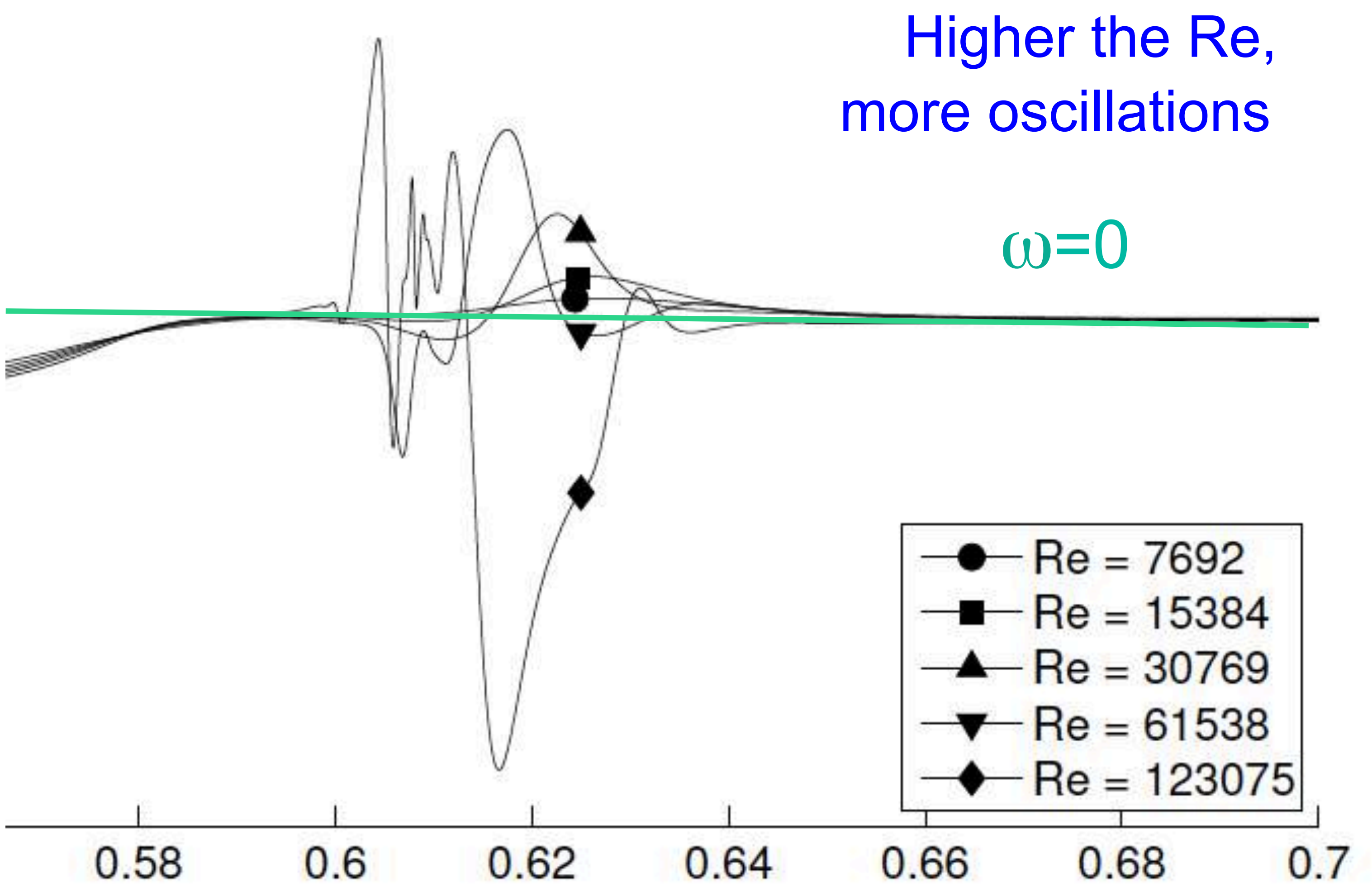


The Prandtl's solution behaves as $k^{-3/2}$ for large k, consistent with the build-up of a jump singularity of vorticity along the wall, while Navier-Stokes develops a bump which spread in k with Re.

Vorticity along the wall at t=57 > t_D

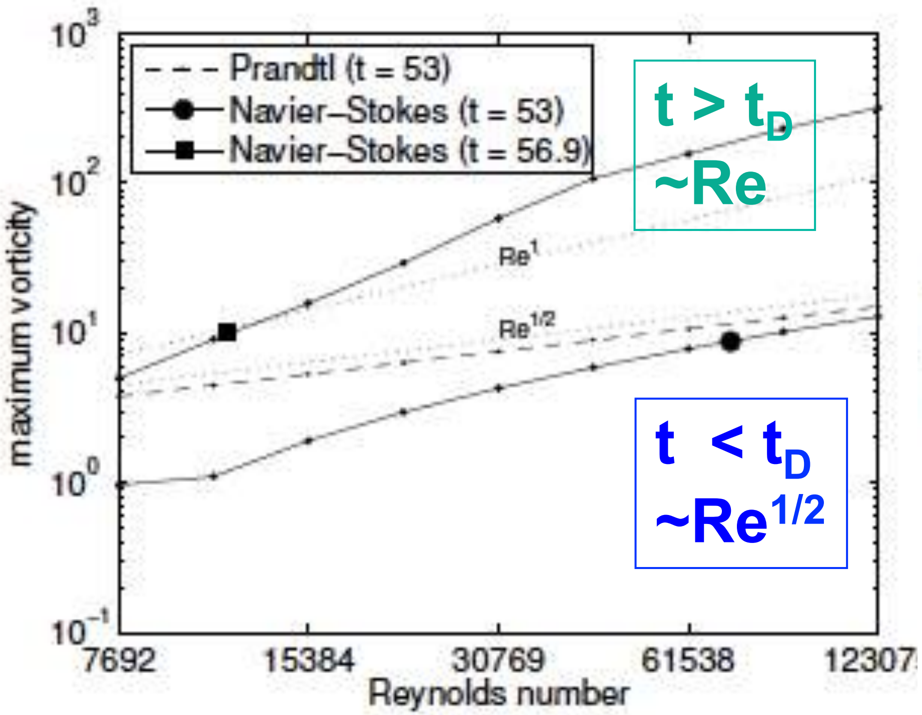


Vorticity along the wall at t=57.5 > t_D

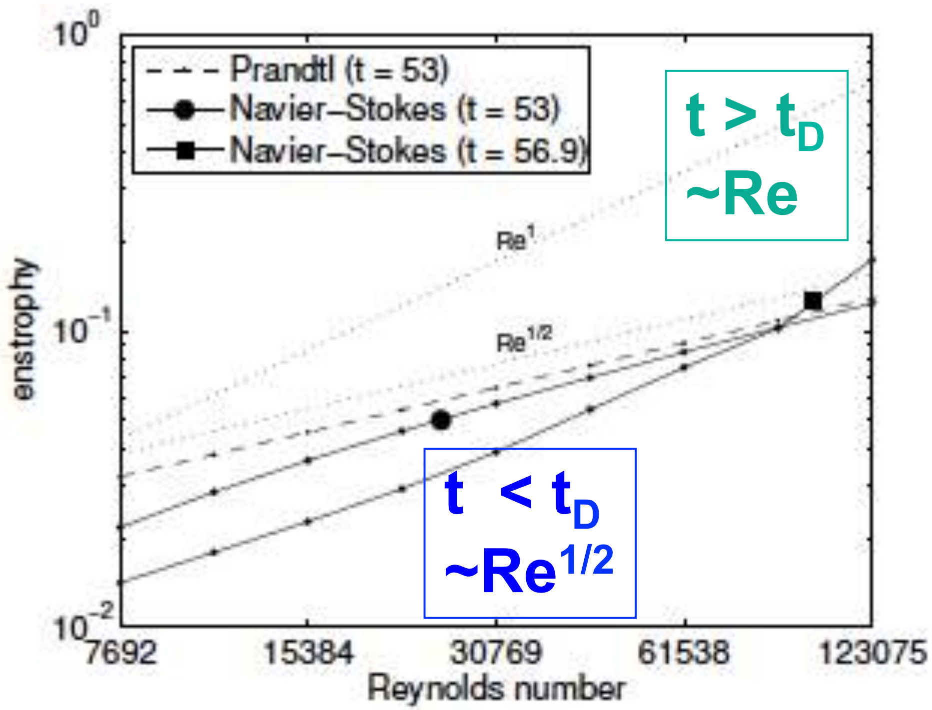


Scaling from Re=7692 to 123075

Vorticity max



Enstrophy



We observe Prandtl's scaling in $Re^{1/2}$ before t_D~ 55.8 and Kato's scaling in Re after.

What about the von Karman log law?

In turbulent boundary layers the mean velocity profile satisfies

$$\langle U(y)\rangle \simeq \frac{U_\tau}{K_{\mathrm{karman}}}\log\left(\frac{yU_\tau}{\nu}\right)$$

the so called log law, where

$$U_\tau = \sqrt{\nu\left\langle \left.\frac{\mathrm{d}U}{\mathrm{d}y}\right|_{y=0}\right\rangle}$$

is the friction velocity

This shows that both the bulk velocity and U_τ have the same scaling with Re (up to a logarithmic factor). This can be seen as a statistical signature of a boundary layer thickness Re^{-1}, which is consistent $\propto$ in some sense with the existence of a Kato layer.

T. von Karman, Uber laminare und turbulente Reibung. Z. ang. Math. Mech. 1 (4), 233{252,, 1921

Conclusions

- The Prandtl solution becomes singular at t_D when BL detaches.
- The Navier-Stokes solution converges uniformly to the Euler solution before BL detaches and ceases to converge after BL detaches.
- The BL detachment involves spatial scales as fine as Re^{-1} produced in different directions, not only parallel to the wall, while attached BL is parallel to the wall and scales as $Re^{-1/2}$.
- The maximal vorticity of Navier-Stokes solution does not appear at the same location of the Prandtl singularity. This contradicts the picture of BL detachment seen as a local process coinciding with Prandtl singularity.

Conclusions

- In regions with reversed flow near the wall, the width of the unstable wavenumber range scales like $Re^{1/2}$, while the amplitude of vorticity continues to scale as $Re^{1/2}$ due to the presence of a Prandtl boundary layer.

- As soon as the buildup of the Prandtl singularity sufficiently excites those wavenumbers, their superposition induces a Re scaling for the amplitude of vorticity.

Conclusions

- By introducing nonlinear Rayleigh-Tollmien-Schlichting waves, followed by roll-up and the injection of a dissipative structure into the bulk flow. However, an essential point to keep in mind is that the phase of these waves is very sensitive to Reynolds.

- In the linear phase, the thickness of the wall-normal sublayer scales like $Re^{-2/3}$, but when the instability becomes nonlinear, vorticity transport induces excitation of scales as fine as Re, leading to dissipation. The process of detachment is thus intricately linked to the occurrence of dissipation.

Conclusions

- The velocity gradient du/dy at the wall scales like Re up to a logarithmic factor, which can be seen as the statistical signature of the existence of a boundary layer of thickness Re in the neighborhood of the wall. Hence, we see that the log-law, as an experimental result, is consistent in some sense with the existence of a Kato layer.

- This connection can be made in a phenomenological way without invoking the Kolmogorov scale and cascade. Our results may help in investigating rigorous foundations to the phenomenological theory of von Karman.

Open questions

Numerical results suggest that a new asymptotic description of the flow beyond the breakdown of the Prandtl regime is possible.

Studying it might help to answer the following questions:

- Would Navier-Stokes solution looses smoothness after t_D?
- Would it converges to a weak singular dissipative solution of Euler's equation analog to dissipative shocks in Burgers solution?
- How can such a weak solution be approximated numerically?

This might lead to a new resolution of d'Alembert's paradox in terms of the production of weak singular dissipative structures due to the interaction of fully-developed turbulent flows with walls.

J. Leray, 1934
Sur le mouvement d'un fluide visqueux,
Acta Mathematica, ***63***

C. de Lellis and L. Székzlyhidi, 2010
Archives Rational Mechanics and Analysis,
195*(1), 221-260*

J. Fluid Mech. (2018), *vol.* 849, *pp.* 676–717.

doi:10.1017/jfm.2018.396

Energy dissipation caused by boundary layer instability at vanishing viscosity

Natacha Nguyen van yen[1], Matthias Waidmann[1], Rupert Klein[1], Marie Farge[2,†] and Kai Schneider[3]

[1]Institut für Mathematik, Freie Universität Berlin, Arnimallee 6, 14195 Berlin, Germany

[2]LMD-CNRS, Ecole Normale Supérieure, 24 rue Lhomond, 75231 Paris CEDEX 5, France

[3]Institut de Mathématiques de Marseille, Aix-Marseille Université and CNRS, Marseille, France

(Received 12 July 2017; revised 4 March 2018; accepted 16 April 2018)